Module F

Markov Analysis

Markov analysis, like decision analysis, is a probabilistic technique. However, Markov analysis is different in that it does not provide a recommended decision. Instead, Markov analysis provides probabilistic information about a decision situation that can aid the decision maker in making a decision. In other words, Markov analysis is not an optimization technique; it is a *descriptive* technique that results in probabilistic information.

Markov analysis is specifically applicable to systems that exhibit probabilistic movement from one state (or condition) to another, over time. For example, Markov analysis can be used to determine the probability that a machine will be running one day and broken down the next or that a customer will change brands of cereal from one month to the next. This latter type of example—referred to as the "brand-switching" problem—will be used to demonstrate the principles of Markov analysis in the following discussion.

The Characteristics of Markov Analysis

The brand-switching problem analyzes the probability of customers' changing brands of a product over time.

Markov analysis can be used to analyze a number of different decision situations; however, one of its most popular applications has been the analysis of customer brand switching. This is basically a marketing application that focuses on the loyalty of customers to a particular product brand, store, or supplier. Markov analysis provides information on the probability of customers' switching from one brand to one or more other brands. An example of the brand-switching problem will be used to demonstrate Markov analysis.

A small community has two gasoline service stations, Petroco and National. The residents of the community purchase gasoline at the two stations on a monthly basis. The marketing department of Petroco surveyed a number of residents and found that the customers were not totally loyal to either brand of gasoline. Customers were willing to change service stations as a result of advertising, service, and other factors. The marketing department found that if a customer bought gasoline from Petroco in any given month, there was only a .60 probability that the customer would buy from Petroco the next month and a .40 probability that the customer would buy gas from National the next month. Likewise, if a customer traded with National in a given month, there was an .80 probability that the customer would purchase gasoline from National in the next month and a .20 probability that the customer would purchase gasoline from Petroco. These probabilities are summarized in Table F-1.

Markov assumptions: (1) the probabilities of moving from a state to all others sum to one, (2) the probabilities apply to all system participants, and (3) the probabilities are constant over time.

This example contains several important assumptions. *First,* notice that in Table F-1 the probabilities in each row sum to one because they are mutually exclusive and collectively exhaustive. This means that if a customer trades with Petroco one month, the customer *must* trade with either Petroco or National the next month (i.e., the customer will not give up buying gasoline, nor will the customer trade with both in one month). *Second,* the probabilities in Table F-1 apply to every customer who purchases gasoline. *Third,* the probabilities in Table F-1 will not change over time. In other words, regardless of when the customer buys gasoline, the probabilities of trading with one of the service stations the next month will be the values in Table F-1. The probabilities in Table F-1 will not change in the future if conditions remain the same.

Table F-1
Probabilities of Customer Movement per Month

	Next Month	
This Month	Petroco	National
Petroco	.60	.40
National	.20	.80

*The **state of the system** is where the system is at a point in time.*

*A **transition probability** is the probability of moving from one state to another during one time period.*

It is these properties that make this example a Markov process. In Markov terminology, the service station a customer trades at in a given month is referred to as a **state of the system**. Thus, this example contains two states of the system—a customer will purchase gasoline at either Petroco or National in any given month. The probabilities of the various states in Table F-1 are known as **transition probabilities**. In other words, they are the probabilities of a customer's making the transition from one state to another during one time period. Table F-1 contains four transition probabilities.

The properties for the service station example just described define a Markov process. They are summarized in Markov terminology as follows:

Summary of Markov properties.

- *Property 1:* The transition probabilities for a given beginning state of the system sum to one.
- *Property 2:* The probabilities apply to all participants in the system.
- *Property 3:* The transition probabilities are constant over time.
- *Property 4:* The states are independent over time.

Markov Analysis Information

Now that we have defined a Markov process and determined that our example exhibits the Markov properties, the next question is, What information will Markov analysis provide? The most obvious information available from Markov analysis is the probability of being in a state at some future time period, which is also the sort of information we can gain from a *decision tree.*

For example, suppose the service stations wanted to know the probability that a customer would trade with it in month 3, given that the customer traded with it this month (1). This analysis can be performed for each service station by using decision trees, as shown in Figures F-1 and F-2.

To determine the probability of a customer's trading with Petroco in month 3, given that the customer initially traded with Petroco in month 1, we must add the two branch probabilities in Figure F-1 associated with Petroco:

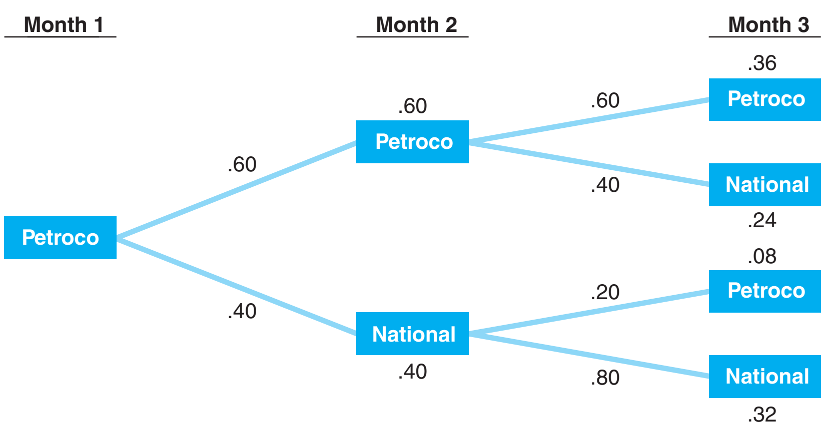
$$.36 + .08 = .44, \text{ the probability of a customer's trading with Petroco in month 3}$$

Likewise, to determine the probability of a customer's purchasing gasoline from National in month 3, we add the two branch probabilities in Figure F-1 associated with National:

$$.24 + .32 = .56, \text{ the probability of a customer's trading with National in month 3}$$

Figure F-1

Probabilities of future states, given that a customer trades with Petroco this month



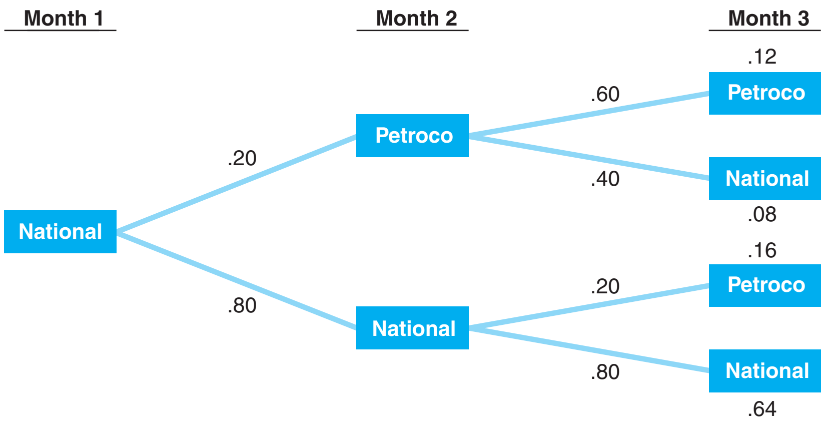


Figure F-2
Probabilities of future states, given that a customer trades with National this month

This same type of analysis can be performed under the condition that a customer initially purchased gasoline from National, as shown in Figure F-2. Given that National is the starting state in month 1, the probability of a customer's purchasing gasoline from National in month 3 is

$$.08 + .64 = .72$$

and the probability of a customer's trading with Petroco in month 3 is

$$.12 + .16 = .28$$

Notice that for each starting state, Petroco and National, the probabilities of ending up in either state in month 3 sum to one:

	Probability of Trade in Month 3		
Starting State	Petroco	National	Sum
Petroco	.44	.56	1.00
National	.28	.72	1.00

The future probabilities of being in a state can be determined by using matrix algebra.

Although the use of decision trees is perfectly logical for this type of analysis, it is time-consuming and cumbersome. For example, if Petroco wanted to know the probability that a customer who traded with it in month 1 will trade with it in month 10, a rather large decision tree would have to be constructed. Alternatively, the same analysis performed previously using decision trees can be done by using *matrix algebra* techniques.

TIME OUT *for Andrey A. Markov*

Andrey Markov, a Russian mathematician, was born in 1856. His early research focused on number theory, which later developed into probability theory. His work focused on the probability of mutually dependent events, and it was in this area that he was able to prove the central limit theorem. He also introduced the concept of chained events that formed the basis for Markov chains and what we now refer to as Markov analysis.

The Transition Matrix

The probabilities of a customer's moving from service station to service station within a 1-month period, presented in tabular form in Table F-1, can also be presented in the form of a rectangular array of numbers called a *matrix,* as follows:

First Month ***Next Month***

$$
T = \begin{array}{r} \\ \text{Petroco} \\ \text{National} \end{array}
\begin{array}{c} \begin{array}{cc} \text{Petroco} & \text{National} \end{array} \\ \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \end{array}
$$

A ***transition matrix*** *includes the transition probabilities for each state of nature.*

Because we previously defined these probabilities as transition probabilities, we will refer to the preceding matrix, T, as a **transition matrix**. The present states of the system are listed on the left of the transition matrix, and the future states in the next time period are listed across the top. For example, there is a .60 probability that a customer who traded with Petroco in month 1 will trade with Petroco in month 2.

Several new symbols are needed for Markov analysis using matrix algebra. We will define the probability of a customer's trading with Petroco in period *i,* given that the customer initially traded with Petroco, as

Probability of trading with Petroco

$$P_{\text{p}}(i)$$

Future period *i*

Initial starting state, Petroco

Similarly, the probability of a customer's trading with National in period *i,* given that a customer initially traded with Petroco, is

Probability of trading with National

$$N_{\text{p}}(i)$$

Future period *i*

Initial starting state, Petroco

For example, the probability of a customer's trading at National in month 2, given that the customer initially traded with Petroco, is

$$N_{\text{p}}(2)$$

The probabilities of a customer's trading with Petroco and National in a future period, *i,* given that the customer traded initially with National, are defined as

$$P_{\text{n}}(i) \quad \text{and} \quad N_{\text{n}}(i)$$

(When interpreting these symbols, always recall that the subscript refers to the starting state.)

Petroco as the initial starting state.

If a customer is currently trading with Petroco (month 1) , the following probabilities exist:

$$P_p(1) = 1.0$$
$$N_p(1) = 0.0$$

In other words, the probability of a customer's trading at Petroco in month 1, given that the customer trades at Petroco, is 1.0.

These probabilities can also be arranged in matrix form, as follows:

$$[P_p(1) \quad N_p(1)] = [1.0 \quad 0.0]$$

Computing probabilities of a customer trading at either station in future months, using matrix multiplication.

This matrix defines the starting conditions of our example system, given that a customer initially trades at Petroco, as in the decision tree in Figure F-1. In other words, a customer is originally trading with Petroco in month 1. We can determine the subsequent probabilities of a customer's trading at Petroco or National in month 2 by multiplying the preceding matrix by the transition matrix, as follows:

$$\begin{aligned} \text{month 2:} \quad [P_p(2) \quad N_p(2)] &= [1.0 \quad 0.0]\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= [.60 \quad .40] \end{aligned}$$

These probabilities of .60 for a customer's trading at Petroco and .40 for a customer's trading at National are the same as those computed in the decision tree in Figure F-1. We use the same procedure for determining the month 3 probabilities, except we now multiply the transition matrix by the month 2 matrix:

$$\begin{aligned} \text{month 3:} \quad [P_p(3) \quad N_p(3)] &= [.60 \quad .40]\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= [.44 \quad .56] \end{aligned}$$

These are the same probabilities we computed by using the decision tree analysis in Figure F-1. However, whereas it would be cumbersome to determine additional values by using the decision tree analysis, we can continue to use the matrix approach as we have previously:

$$\begin{aligned} \text{month 4:} \quad [P_p(4) \quad N_p(4)] &= [.44 \quad .56]\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= [.38 \quad .62] \end{aligned}$$

The state probabilities for several subsequent months are as follows:

$$\begin{aligned} &\text{month 5:} \quad [P_p(5) \quad N_p(5) = [.35 \quad .65] \\ &\text{month 6:} \quad [P_p(6) \quad N_p(6) = [.34 \quad .66] \\ &\text{month 7:} \quad [P_p(7) \quad N_p(7) = [.34 \quad .66] \\ &\text{month 8:} \quad [P_p(8) \quad N_p(8) = [.33 \quad .67] \\ &\text{month 9:} \quad [P_p(9) \quad N_p(9) = [.33 \quad .67] \end{aligned}$$

In future periods, the state probabilities become constant.

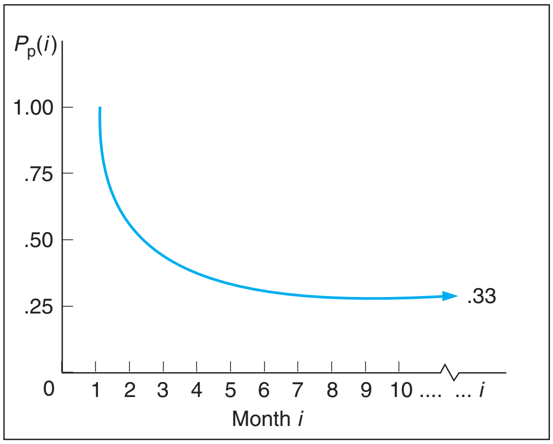
Notice that as we go farther and farther into the future, the changes in the state probabilities become smaller and smaller, until eventually there are no changes at all. At that point every month in the future will have the same probabilities. For this example, the state probabilities that result after some future month, *i*, are

$$[P_p(i) \quad N_p(i)] = [.33 \quad .67]$$

This characteristic of the state probabilities approaching a constant value after a number of time periods is shown for $P_p(i)$ in Figure F-3.

Figure F-3

The probability $P_p(i)$ for future values of i



$N_p(i)$ exhibits this same characteristic as it approaches a value of .67. This is a potentially valuable result for the decision maker. In other words, the service station owner can now conclude that after a certain number of months in the future, there is a .33 probability that the customer will trade with Petroco if the customer initially traded with Petroco.

Computing future state probabilities when the initial starting state is National

This same type of analysis can be performed given the starting condition in which a customer initially trades with National in month 1. This analysis, shown as follows, corresponds to the decision tree in Figure F-2.

Given that a customer initially trades at the National station, then

$$[P_n(1) \quad N_n(1)] = [0.0 \quad 1.0]$$

Using these initial starting-state probabilities, we can compute future-state probabilities as follows:

$$\text{month 2:} \quad [P_n(2) \quad N_n(2)] = [0.0 \quad 1.0]\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}$$
$$= [.20 \quad .80]$$

$$\text{month 3:} \quad [P_n(3) \quad N_n(3)] = [.20 \quad .80]\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}$$
$$= [.28 \quad .72]$$

These are the same values obtained by using the decision tree analysis in Figure F-2. Subsequent state probabilities, computed similarly, are shown next:

$$\text{month 4:} \quad [P_n(4) \quad N_n(4)] = [.31 \quad .69]$$
$$\text{month 5:} \quad [P_n(5) \quad N_n(5)] = [.32 \quad .68]$$
$$\text{month 6:} \quad [P_n(6) \quad N_n(6)] = [.33 \quad .67]$$
$$\text{month 7:} \quad [P_n(7) \quad N_n(7)] = [.33 \quad .67]$$
$$\text{month 8:} \quad [P_n(8) \quad N_n(8)] = [.33 \quad .67]$$
$$\text{month 9:} \quad [P_n(9) \quad N_n(9)] = [.33 \quad .67]$$

The probability of ending up in a state in the future is independent of the starting state.

As in the previous case in which Petroco was the starting state, these state probabilities also become constant after several periods. However, notice that the eventual state probabilities (i.e., .33 and .67) achieved when National is the starting state *are exactly the same* as the previous state probabilities achieved when Petroco was the starting state. In other words, the probability of ending up in a particular state in the future is not dependent on the starting state.

Steady-State Probabilities

***Steady-state probabilities** are average, constant probabilities that the system will be in a state in the future.*

The probabilities of .33 and .67 in our example are referred to as **steady-state probabilities**. The steady-state probabilities are average probabilities that the system will be in a certain state after a large number of transition periods. This does not mean the system stays in one state. The system will continue to move from state to state in future time periods; however, the average *probabilities* of moving from state to state for all periods will remain constant in the long run. In a Markov process, after a number of periods have passed, the probabilities will approach steady state.

For our service station example, the steady-state probabilities are

.33 = probability of a customer's trading at Petroco after a number of months in the future, regardless of where the customer traded in month 1

.67 = probability of a customer's trading at National after a number of months in the future, regardless of where the customer traded in month 1

Notice that in the determination of the preceding steady-state probabilities, we considered each starting state separately. First, we assumed that a customer was initially trading at Petroco, and the steady-state probabilities were computed given this starting condition. Then we determined that the steady-state probabilities were the same, regardless of the starting condition. However, it was not necessary to perform these matrix operations separately. We could have simply combined the operations into one matrix, as follows:

$$\text{month 2:}\quad \begin{bmatrix} P_p(2) & N_p(2) \\ P_n(2) & N_n(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} = \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}$$

$$\text{month 3:}\quad \begin{bmatrix} P_p(3) & N_p(3) \\ P_n(3) & N_n(3) \end{bmatrix} = \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} = \begin{bmatrix} .44 & .56 \\ .28 & .72 \end{bmatrix}$$

$$\text{month 4:}\quad \begin{bmatrix} P_p(4) & N_p(4) \\ P_n(4) & N_n(4) \end{bmatrix} = \begin{bmatrix} .44 & .56 \\ .28 & .72 \end{bmatrix}\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} = \begin{bmatrix} .38 & .62 \\ .31 & .69 \end{bmatrix}$$

until eventually we arrived at the steady-state probabilities:

$$\text{month 9:}\quad \begin{bmatrix} P_p(9) & N_p(9) \\ P_n(9) & N_n(9) \end{bmatrix} = \begin{bmatrix} .33 & .67 \\ .33 & .67 \end{bmatrix}$$

Direct Algebraic Determination of Steady-State Probabilities

In the previous section, we computed the state probabilities for approximately eight periods (i.e., months) before the steady-state probabilities were reached for both states. This required quite a few matrix computations. Alternatively, it is possible to solve for the steady-state probabilities directly, without going through all these matrix operations.

At some point in the future, the state probabilities remain constant from period to period.

Notice that after eight periods in our previous analysis, the state probabilities did not change from period to period (i.e., from month to month). For example,

$$\begin{aligned}\text{month 8:}\quad [P_p(8) \quad N_p(8)] &= [.33 \quad .67]\\ \text{month 9:}\quad [P_p(9) \quad N_p(9)] &= [.33 \quad .67]\end{aligned}$$

Thus, we can also say that after a number of periods in the future (in this case, eight), the state probabilities in period i equal the state probabilities in period $i + 1$. For our example, this means that

$$[P_p(8) \quad N_p(8)] = [P_p(9) \quad N_p(9)]$$

After steady state is reached, it is not necessary to designate the time period.

In fact, it is not necessary to designate which period in the future is actually occurring. That is,

$$[P_p \quad N_p] = [P_p \quad N_p]$$

given steady-state conditions.

These probabilities are for some period, i, in the future once a steady state has already been reached. To determine the state probabilities for period $i + 1$, we would normally do the following computation:

$$[P_p(i + 1) \quad N_p(i + 1)] = [P_p(i) \quad N_p(i)]\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}$$

However, we have already stated that once a steady state has been reached, then

$$[P_p(i + 1) \quad N_p(i + 1)] = [P_p(i) \quad N_p(i)]$$

and it is not necessary to designate the period. Thus, our computation can be rewritten as

$$[P_p \quad N_p] = [P_p \quad N_p]\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}$$

Steady-state probabilities can be computed by developing a set of equations, using matrix operations, and solving them simultaneously.

Performing matrix operations results in the following set of equations:

$$\begin{aligned} P_p &= .6P_p + .2N_p \\ N_p &= .4P_p + .8N_p \end{aligned}$$

Recall that the transition probabilities for a row in the transition matrix (i.e., the state probabilities) must sum to one:

$$P_p + N_p = 1.0$$

which can also be written as

$$N_p = 1.0 - P_p$$

Substituting this value into our first foregoing equation ($P_p = .6P_p + .2N_p$) results in the following:

$$\begin{aligned} P_p &= .6P_p + .2(1.0 - P_p) = .6P_p + .2 - .2P_p = .2 + .4P_p \\ .6P_p &= .2 \\ P_p &= .2/.6 = .33 \end{aligned}$$

and

$$N_p = 1.0 - P_p = 1.0 - .33 = .67$$

These are the steady-state probabilities we computed in our previous analysis:

$$[P_p \quad N_p] = [.33 \quad .67]$$

Application of the Steady-State Probabilities

Steady-state probabilities can be multiplied by the total system participants to determine the expected number in each state in the future.

The steady-state probabilities indicate not only the probability of a customer's trading at a particular service station in the long-term future but also the *percentage of customers* who will trade at a service station during any given month in the long run. For example, if there are 3,000 customers in the community who purchase gasoline, then in the long run the following *expected* number will purchase gasoline at each station on a monthly basis:

$$\begin{aligned} \text{Petroco:} \quad P_p(3{,}000) &= .33(3{,}000) \\ &= 990 \text{ customers} \\ \text{National:} \quad N_p(3{,}000) &= .67(3{,}000) \\ &= 2{,}010 \text{ customers} \end{aligned}$$

Now suppose that Petroco has decided that it is getting less than a reasonable share of the market and would like to increase its market share. To accomplish this objective, Petroco has improved its service substantially, and a survey indicates that the transition probabilities have changed to the following:

$$T = \begin{array}{r} \\ \text{Petroco} \\ \text{National} \end{array} \begin{array}{c} \begin{array}{cc} \text{Petroco} & \text{National} \end{array} \\ \begin{bmatrix} .70 & .30 \\ .20 & .80 \end{bmatrix} \end{array}$$

In other words, the improved service has resulted in a smaller probability (.30) that customers who traded initially at Petroco will switch to National the next month.

Now we will recompute the steady-state probabilities, based on this new transition matrix:

$$[P_p \quad N_p] = [P_p \quad N_p] \begin{bmatrix} .70 & .30 \\ .20 & .80 \end{bmatrix}$$

$$\begin{aligned} P_p &= .7P_p + .2N_p \\ N_p &= .3P_p + .8N_p \end{aligned}$$

Using the first equation and the fact that $N_p = 1.0 - P_p$, we have

$$\begin{aligned} P_p &= .7P_p + .2(1.0 - P_p) = .7P_p + .2 - .2P_p \\ .5P_p &= .2 \\ P_p &= .2/.5 = .4 \end{aligned}$$

and thus

$$N_p = 1 - P_p = 1 - .4 = .6$$

This means that out of the 3,000 customers, Petroco will now get 1,200 customers (i.e., $.40 \times 3{,}000$) in any given month in the long run. Thus, improvement in service will result in an increase of 210 customers per month (if the new transition probabilities remain constant for a long period of time in the future). In this situation Petroco must evaluate the trade-off between the cost of the improved service and the increase in profit from the additional 210 customers. For example, if the improved service costs $1,000 per month, then the extra 210 customers must generate an increase in profit greater than $1,000 to justify the decision to improve service.

This brief example demonstrates the usefulness of Markov analysis for decision making. Although Markov analysis does not yield a recommended decision (i.e., a solution), it does provide information that will help the decision maker to make a decision.

Determination of Steady States with QM for Windows

QM for Windows has a Markov analysis module, which is extremely useful when the dimensions of the transition matrix exceed two states. The algebraic computations required to determine steady-state probabilities for a transition matrix with even three states are lengthy; for a matrix with more than three states, computing capabilities are a necessity. Markov analysis with QM for Windows will be demonstrated using the service station example in this section.

Exhibit F-1 shows our example input data for the Markov analysis module in QM for Windows. Note that it is not necessary to enter a number of transitions to get the steady-state probabilities. The program automatically computes the steady state. "Number of Transitions" refers to the number of transition computations you might like to see.

Exhibit F-1

Number of transitions: 0

Instruction: Enter either the probability of starting in petroco or the number of elements that start in petroco. Any non-negative value is permissible.

Market Share Analysis

	Initial	Petroco	National
Petroco	0	.6	.4
National	0	.2	.8

Exhibit F-2 shows the solution with the steady-state transition matrix for our service station example.

Exhibit F-2

Markov Analysis Results

Market Share Analysis Steady state transition matrix

	Petroco	National
Petroco	.3334	.6667
National	.3334	.6667
Ending number (given initial)	0	0
Steady State probability	.3334	.6667

Additional Examples of Markov Analysis

A machine breakdown example.

Although analyzing brand switching is probably the most popular example of Markov analysis, this technique does have other applications. One prominent application relates to the breakdown of a machine or system (such as a computer system, a production operation, or an electrical system). For example, a particular production machine could be assigned the states "operating" and "breakdown." The transition probabilities could then reflect the probability of a machine's either breaking down or operating in the next time period (i.e., month, day, or year).

As an example, consider a machine that has the following daily transition matrix:

$$\begin{array}{ll} \textbf{\textit{Day 1}} & \qquad\qquad \textbf{\textit{Day 2}} \\ & \begin{array}{cc} \text{Operate} & \text{Breakdown} \end{array} \\ T = \begin{array}{l} \text{Operate} \\ \text{Breakdown} \end{array} & \begin{bmatrix} .90 & .10 \\ .70 & .30 \end{bmatrix} \end{array}$$

The steady-state probabilities for this example are

$$\begin{aligned} .88 &= \text{steady-state probability of the machine's operating} \\ .12 &= \text{steady-state probability of the machine's breaking down} \end{aligned}$$

Now if management decides that the long-run probability of .12 for a breakdown is excessive, it might consider increasing preventive maintenance, which would change the transition matrix for this example. The decision to increase maintenance would be based on the cost of the increase versus the value of the increased production output gained from having fewer breakdowns.

A rental truck firm with three states.

Thus far in our discussion of Markov analysis, we have considered only examples that consisted of two states. This has been partially a matter of convenience because 2×2 matrices are easier to work with than matrices of a higher magnitude. However, examples that contain a larger number of states are analyzed in the same way as our previous examples. For example, consider the Carry-All Rental Truck Firm, which serves three states—Virginia, North Carolina, and Maryland. Trucks are rented on a daily basis and can be rented and returned in any of the three states. The transition matrix for this example follows:

$$\begin{array}{ll} \textbf{\textit{Rented}} & \qquad\qquad \textbf{\textit{Returned}} \\ & \begin{array}{ccc} \text{Virginia} & \text{Maryland} & \text{North Carolina} \end{array} \\ T = \begin{array}{l} \text{Virginia} \\ \text{Maryland} \\ \text{North Carolina} \end{array} & \begin{bmatrix} .60 & .20 & .20 \\ .30 & .50 & .20 \\ .40 & .10 & .50 \end{bmatrix} \end{array}$$

The steady-state probabilities for this example are determined by using the same algebraic approach presented earlier, although the mathematical steps are more lengthy and complex. Instead of solving three simultaneous equations, we solve four.

The steady-state probabilities for this example are

$$\begin{array}{ccc} \text{Virginia} & \text{Maryland} & \text{North Carolina} \end{array}$$
$$\begin{bmatrix} .471 & .244 & .285 \end{bmatrix}$$

Thus, in the long run, these percentages of Carry-All trucks will end up in the three states. If the company had 200 trucks, then it could expect to have the following number of trucks available in each state at any time in the future:

$$\begin{array}{ccc} \text{Virginia} & \text{Maryland} & \text{North Carolina} \end{array}$$
$$\begin{bmatrix} 94 & 49 & 57 \end{bmatrix}$$

Special Types of Transition Matrices

In some cases the transition matrix derived from a Markov problem is not in the same form as those in the examples shown in this module. Some matrices have certain characteristics that alter the normal methods of Markov analysis. Although a detailed analysis of these special cases is beyond the scope of this module, we will give examples of them so that they can be easily recognized.

In the transition matrix

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .40 & .60 & 0 \\ .30 & .70 & 0 \\ 1.0 & 0 & 0 \end{bmatrix} \end{array}$$

Once the system leaves a transient state, it will never return.

state 3 is a *transient state.* Once state 3 is achieved, the system will never return to that state. Both states 1 and 2 contain a 0.0 probability of going to state 3. The system will move out of state 3 to state 1 (with a 1.0 probability) but will never return to state 3.

A transition matrix is cyclic when the system moves back and forth between states.

The following transition matrix is referred to as cyclic:

$$T = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1.0 \\ 1.0 & 0 \end{bmatrix} \end{array}$$

The system will simply cycle back and forth between states 1 and 2, without ever moving out of the cycle.

Finally, consider the following transition matrix for states 1, 2, and 3:

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .30 & .60 & .10 \\ .40 & .40 & .20 \\ 0 & 0 & 1.0 \end{bmatrix} \end{array}$$

Once the system moves into an absorbing state, it is trapped and cannot leave.

State 3 in this transition matrix is referred to as an *absorbing,* or trapping, state. Once state 3 is achieved, there is a 1.0 probability that it will be achieved in succeeding time periods. Thus, the system in effect ends when state 3 is achieved. There is no movement from an absorbing state; the item is trapped in that state.

The Debt Example

A unique and popular application of an absorbing state matrix is the bad debt example. In this example, the states are the months during which a customer carries a debt. The customer may pay the debt (i.e., a bill) at any time and thus achieve an absorbing state for payment. However, if the customer carries the debt longer than a specified number of periods (say, 2 months), the debt will be labeled "bad" and will be transferred to a bill collector. The state "bad debt" is also an absorbing state. Through various matrix manipulations, the portion of accounts receivable that will be paid and the portion that will become bad debts can be determined. (Because of these matrix manipulations, the debt example is somewhat more complex than the Markov examples presented previously.)

The debt example will be demonstrated by using the following transition matrix, which describes the accounts receivable for the A-to-Z Office Supply Company:

$$T = \begin{array}{c} \\ p \\ 1 \\ 2 \\ b \end{array} \begin{array}{c} \begin{array}{cccc} p & 1 & 2 & b \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .70 & 0 & .30 & 0 \\ .50 & 0 & 0 & .50 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

An absorbing state has a transition probability of one.

In this absorbing state transition matrix, state p indicates that a debt is paid, states 1 and 2 indicate that a debt is 1 or 2 months old, respectively, and state b indicates that a debt becomes bad. Notice that once a debt is paid (i.e., once the item enters state p), the probability of moving to state 1, 2, or b is zero. If the debt is 1 month old, there is a .70 probability that it will be paid in the next month and a .30 probability that it will go to month 2 unpaid. If the debt is in month 2, there is a .50 probability that it will be paid and a .50 probability that it will become a bad debt in the next time period. Finally, if the debt is bad, there is a zero probability that it will return to any previous state.

The next step in analyzing this Markov problem is to rearrange the transition matrix into the following form:

$$T = \begin{array}{c} \\ p \\ b \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} p & b & 1 & 2 \end{array} \\ \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline .70 & 0 & 0 & .30 \\ .50 & .50 & 0 & 0 \end{array}\right] \end{array}$$

We have now divided the transition matrix into four parts, or submatrices, which we label as follows:

$$T = \left[\begin{array}{c:c} I & 0 \\ \hdashline R & Q \end{array}\right]$$

where

$$I = \begin{array}{c} \\ p \\ b \end{array} \begin{array}{c} \begin{array}{cc} p & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

= an identity matrix

$$0 = \begin{array}{c} \\ p \\ b \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

= a matrix of zeros

$$R = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} p & b \end{array} \\ \begin{bmatrix} .70 & 0 \\ .50 & .50 \end{bmatrix} \end{array}$$

= a matrix containing the transition probabilities of the debt's being absorbed in the next period

$$Q = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & .30 \\ 0 & 0 \end{bmatrix} \end{array}$$

= a matrix containing the transition probabilities for movement between both nonabsorbing states

The matrix labeled I is an *identity matrix,* so called because it has ones along the diagonal and zeros elsewhere in the matrix.

The first matrix operation to be performed determines the *fundamental matrix, F,* as follows:

$$F = (I - Q)^{-1}$$

The notation to raise the $(I - Q)$ matrix to the -1 power indicates what is referred to as the *inverse* of a matrix. The fundamental matrix is computed by taking the inverse of the difference between the identity matrix, I, and Q. For our example, the fundamental matrix is computed as follows:

$$\begin{aligned} F &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & .30 \\ 0 & 0 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} 1 & -.30 \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & .30 \\ 0 & 1 \end{bmatrix} \end{array} \end{aligned}$$

The fundamental matrix indicates the expected number of times the system will be in any of the nonabsorbing states before absorption occurs (for our example, before a debt becomes bad or is paid). Thus, according to F, if the customer is in state 1 (1 month late in paying the debt), the expected number of times the customer will be 2 months late will be .30 before the debt is paid or becomes bad.

Next we will multiply the fundamental matrix by the R matrix created when the original transition matrix was partitioned:

$$\begin{aligned} F \times R &= \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & .30 \\ 0 & 1 \end{bmatrix} \end{array} \times \begin{array}{c c} & \begin{array}{cc} p & b \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} .70 & 0 \\ .50 & .50 \end{bmatrix} \end{array} \\ &= \begin{array}{c c} & \begin{array}{cc} p & b \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} .85 & .15 \\ .50 & .50 \end{bmatrix} \end{array} \end{aligned}$$

The $F \times R$ matrix reflects the probability that the debt will eventually be absorbed, given any starting state. For example, if the debt is presently in the first month, there is an .85 probability that it will eventually be paid and a .15 probability that it will result in a bad debt.

Now, suppose the A-to-Z Office Supply Company has accounts receivable of \$4,000 in month 1 and \$6,000 in month 2. To determine what portion of these funds will be collected and what portion will result in bad debts, we multiply a matrix of these dollar amounts by the $F \times R$ matrix:

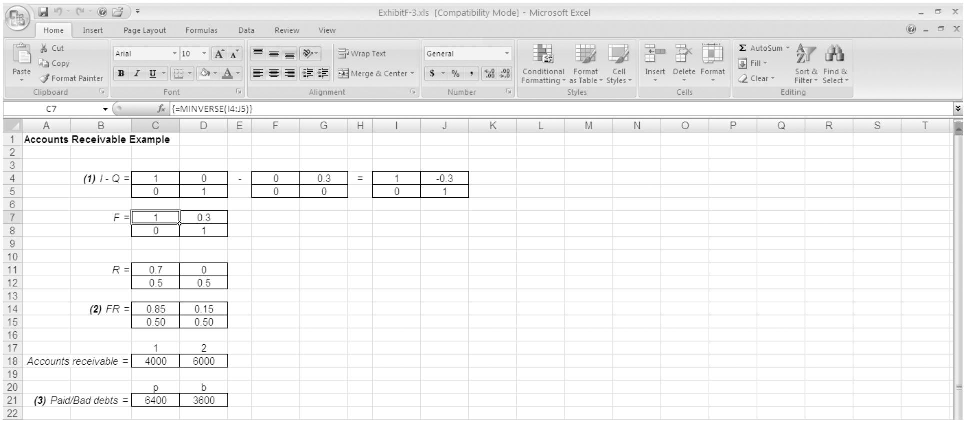
$$\begin{aligned} \text{determination of accounts receivable} &= \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 4{,}000 & 6{,}000 \end{bmatrix} \end{array} \times \begin{array}{c c} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} .85 & .15 \\ .50 & .50 \end{bmatrix} \end{array} \\ &= \begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \begin{bmatrix} 6{,}400 & 3{,}600 \end{bmatrix} \end{array} \end{aligned}$$

Thus, of the total \$ 10,000 owed, the office supply company can expect to receive \$6,400, and \$3,600 will become bad debts. The debt example can be analyzed even further than we have done here, although the mathematics become increasingly difficult.

Excel Solution of the Debt Example

All the matrix operations performed manually in the previous section for the debt example can be accomplished by using Excel. Exhibit F-3 shows an Excel spreadsheet with the matrix operations for our A-to-Z Office Supply Company accounts receivable example.

Exhibit F-3



In step 1 the fundamental matrix, *F*, is developed. First, the *Q* matrix is subtracted from the identity matrix, *I*. This can be done by inputting the numeric values from our example in the matrices set up in cells **C4:D5** and **F4:G5**, and subtracting the array in **F4:G5** from the array in **C4:D5**. This subtraction can be conveniently accomplished by covering cells **I4:J5** with the cursor, embedding the formula **=(C4:D5) − (F4:G5)** in cell **I4**; then with the "Ctrl" and "Shift" keys pressed down, we press "Enter."

Next, we take the inverse of the matrix in cells **I4:J5**. This is done by covering cells **C7:D8** with the cursor and entering the formula shown on the formula bar at the top of the spreadsheet in cell C7. Then with the "Ctrl" and "Shift" keys pressed down, we press "Enter."

In step 2 the *FR* matrix is computed by entering the matrix operation formulas for multiplying two matrices in cells **C14:D15**. For example, the formula **=C7*C11+D7*C12** is entered in cell C14, which results in the value 0.85 in cell C14.

Finally, in step 3 the matrix values indicating the amounts of paid and bad debts for our example are computed, using the formulas for multiplying two matrices. For example, the formula in cell C21 is **=C14*C18+C15*D18,** resulting in the value 6,400.

Problems

1. A town has three gasoline stations, Petroco, National, and Gascorp. The residents purchase gasoline on a monthly basis. The following transition matrix contains the probabilities of the customers' purchasing a given brand of gasoline next month:

This Month ***Next Month***

$$\begin{array}{l} \\ \text{Petroco} \\ \text{National} \\ \text{Gascorp} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Petroco} & \text{National} & \text{Gascorp} \end{array} \\ \begin{bmatrix} .5 & .3 & .2 \\ .1 & .7 & .2 \\ .1 & .1 & .8 \end{bmatrix} \end{array}$$

Using a decision tree, determine the probabilities of a customer's purchasing each brand of gasoline in month 3, given that the customer purchases National in the present month. Summarize the resulting probabilities in a table.

2. Discuss the properties that must exist for the transition matrix in Problem 1 to be considered a Markov process.

3. The only grocery store in a community stocks milk from two dairies—Creamwood and Cheesedale. The following transition matrix shows the probabilities of a customer's purchasing each brand of milk next week, given that he or she purchased a particular brand this week:

$$\begin{array}{l c}
\textbf{\textit{This Week}} & \textbf{\textit{Next Week}} \\
 & \begin{array}{cc} \text{Creamwood} & \text{Cheesedale} \end{array} \\
\begin{array}{l} \text{Creamwood} \\ \text{Cheesedale} \end{array} & \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}
\end{array}$$

Given that a customer purchases Creamwood milk this week, use a decision tree to determine the probability that he or she will purchase Cheesedale milk in week 4.

4. Given the transition matrix in Problem 1, use matrix multiplication methods to determine the state probabilities for month 3, given that a customer initially purchases Petroco gasoline.

5. Determine the state probabilities in Problem 3 by using matrix multiplication methods.

6. A manufacturing firm has developed a transition matrix containing the probabilities that a particular machine will operate or break down in the following week, given its operating condition in the present week:

$$\begin{array}{l c}
\textbf{\textit{This Week}} & \textbf{\textit{Next Week}} \\
 & \begin{array}{cc} \text{Operate} & \text{Break Down} \end{array} \\
\begin{array}{l} \text{Operate} \\ \text{Break Down} \end{array} & \begin{bmatrix} .4 & .6 \\ .8 & .2 \end{bmatrix}
\end{array}$$

   a. Assuming that the machine is operating in week 1, determine the probabilities that the machine will operate or break down in week 2, week 3, week 4, week 5, and week 6.
   b. Determine the steady-state probabilities for this transition matrix algebraically and indicate the percentage of future weeks in which the machine will break down.

7. A city is served by two newspapers—the *Tribune* and the *Daily News*. Each Sunday readers purchase one of the newspapers at a stand. The following transition matrix contains the probabilities of a customer's buying a particular newspaper in a week, given the newspaper purchased the previous Sunday:

$$\begin{array}{l c}
\textbf{\textit{This Sunday}} & \textbf{\textit{Next Sunday}} \\
 & \begin{array}{cc} \textit{Tribune} & \textit{Daily News} \end{array} \\
\begin{array}{l} \textit{Tribune} \\ \textit{Daily News} \end{array} & \begin{bmatrix} .65 & .35 \\ .45 & .55 \end{bmatrix}
\end{array}$$

Determine the steady-state probabilities for this transition matrix algebraically and explain what they mean.

8. The Hergeshiemer Department Store wants to analyze the payment behavior of customers who have outstanding accounts. The store's credit department has determined the following bill payment pattern for credit customers from historical records:

$$\begin{array}{l c}
\textbf{\textit{Present Month}} & \textbf{\textit{Next Month}} \\
 & \begin{array}{cc} \text{Pay} & \text{Not Pay} \end{array} \\
\begin{array}{l} \text{Pay} \\ \text{Not Pay} \end{array} & \begin{bmatrix} .9 & .1 \\ .8 & .2 \end{bmatrix}
\end{array}$$

   a. If a customer did not pay his or her bill in the present month, what is the probability that the bill will not be paid in any of the next 3 months?
   b. Determine the steady-state probabilities for this transition matrix and explain what they mean.

9. A rural community has two television stations, and each Wednesday night the local viewers watch either the *Wednesday Movie* or a show called *Western Times*. The following transition matrix contains the probabilities of a viewer's watching one of the shows in a week, given that he or she watched a particular show the preceding week:

$$\begin{array}{l c}
\textbf{\textit{This Week}} & \textbf{\textit{Next Week}} \\
 & \begin{array}{cc} \text{Movie} & \text{Western} \end{array} \\
\begin{array}{l} \text{Movie} \\ \text{Western} \end{array} & \begin{bmatrix} .75 & .25 \\ .45 & .55 \end{bmatrix}
\end{array}$$

   a. Determine the steady-state probabilities for this transition matrix algebraically.
   b. If the community contains 1,200 television sets, how many will be tuned to each show in the long run?
   c. If a prospective local sponsor wanted to pay for commercial time on one of the shows, which show would more likely be selected?

10. In Problem 3, assume that 600 gallons of milk are sold weekly, regardless of the brand purchased.
   a. How many gallons of each brand of milk will be purchased in any given week in the long run?
   b. The Cheesedale dairy is considering paying \$500 per week for a new advertising campaign that would alter the brand-switching probabilities as follows:

$$\begin{array}{l c}
\textbf{\textit{This Week}} & \textbf{\textit{Next Week}} \\
 & \begin{array}{cc} \text{Creamwood} & \text{Cheesedale} \end{array} \\
\begin{array}{l} \text{Creamwood} \\ \text{Cheesedale} \end{array} & \begin{bmatrix} .6 & .4 \\ .2 & .8 \end{bmatrix}
\end{array}$$

   If each gallon of milk sold results in \$1.00 in profit for Cheesedale, should the dairy institute the advertising campaign?

11. The manufacturing company in Problem 6 is considering a preventive maintenance program that would change the operating probabilities as follows:

$$\begin{array}{l c}
\textbf{\textit{This Week}} & \textbf{\textit{Next Week}} \\
 & \begin{array}{cc} \text{Operate} & \text{Break Down} \end{array} \\
\begin{array}{l} \text{Operate} \\ \text{Break Down} \end{array} & \begin{bmatrix} .7 & .3 \\ .9 & .1 \end{bmatrix}
\end{array}$$

   The machine earns the company \$1,000 in profit each week it operates. The preventive maintenance program would cost \$8,000 per year. Should the company institute the preventive maintenance program?

12. In Problem 7, assume that 20,000 newspapers are sold each Sunday, regardless of the publisher.
    a. How many copies of the *Tribune* and the *Daily News* will be purchased in a given week in the long run?
    b. The *Daily News* is considering a promotional campaign estimated to change the weekly reader probabilities as follows:

$$\begin{array}{l l} \textbf{\textit{This Sunday}} & \textbf{\textit{Next Sunday}} \\ & \begin{array}{cc} \textit{Tribune} & \textit{Daily News} \end{array} \\ \begin{array}{l} \textit{Tribune} \\ \textit{Daily News} \end{array} & \begin{bmatrix} .5 & .5 \\ .3 & .7 \end{bmatrix} \end{array}$$

The promotional campaign will cost \$150 per week. Each newspaper sold earns the *Daily News* \$0.05 in profit. Should the paper adopt the promotional campaign?

13. The following transition matrix describes the accounts receivable process for the Ewing-Barnes Department Store:

$$T = \begin{array}{c} \\ \mathrm{p} \\ 1 \\ 2 \\ \mathrm{b} \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{p} & 1 & 2 & \mathrm{b} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .80 & 0 & .20 & 0 \\ .40 & 0 & 0 & .60 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The states p and b represent an account that is paid and a bad account, respectively. The numbers 1 and 2 represent the fact that an account is either 1 or 2 months overdue, respectively. After an account has been overdue for 2 months, it becomes a bad account and is transferred to the store's overdue accounts section for collection. The company has sales of \$210,000 each month. Determine how much the company will be paid and how many of the debts will become bad debts in a 2-month period.

14. The department store in Problem 13 will never be able to collect 20% of the bad accounts, and it costs the store an additional \$0.10 per dollar collected to collect the remaining bad accounts. The store management is contemplating a new, more restrictive credit plan that would reduce sales to an estimated \$195,000 per month. However, the tougher credit plan would result in the following transition matrix for accounts receivable:

$$T = \begin{array}{c} \\ \mathrm{p} \\ 1 \\ 2 \\ \mathrm{b} \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{p} & 1 & 2 & \mathrm{b} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .90 & 0 & .10 & 0 \\ .70 & 0 & 0 & .30 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Determine whether the store should adopt the more restrictive credit plan or keep the existing one.

15. In Westvale, a small, rural town in Maine, virtually all shopping and business are done in the town. The town has one farm and garden center that sells fertilizer to the local farmers and gardeners. The center carries three brands of fertilizer—Plant Plus, Crop Extra, and Gro-fast—and every person in the town who uses fertilizer uses one of the three brands. The garden center has 9,000 customers for fertilizer each spring. An extensive market research study has determined that customers switch brands of fertilizer according to the following probability transition matrix:

$$\begin{array}{l l} \textbf{\textit{This Spring}} & \textbf{\textit{Next Spring}} \\ & \begin{array}{ccc} \text{Plant Plus} & \text{Crop Extra} & \text{Gro-fast} \end{array} \\ \begin{array}{l} \text{Plant Plus} \\ \text{Crop Extra} \\ \text{Gro-fast} \end{array} & \begin{bmatrix} .4 & .3 & .3 \\ .5 & .1 & .4 \\ .4 & .2 & .4 \end{bmatrix} \end{array}$$

The number of customers currently using each brand of fertilizer is shown in the following table:

Fertilizer Brand	Customers
Plant Plus	3,000
Crop Extra	4,000
Gro-fast	2,000

a. Determine the steady-state probabilities for the fertilizer brands.
b. Forecast the customer demand for each brand of fertilizer in the long run and the changes in customer demand.

16. Westvale, a small community in Maine, has 7,000 bank patrons that do their banking business at one of three banks in town, the American National Bank, the Bank of Westvale, and the Commerce Bank. The following transition matrix shows the probability that a bank customer will trade with the same bank next month or move to one of the other banks.

Present Month ***Next Month***

$$\begin{array}{c c} & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} .80 & .10 & .10 \\ .10 & .70 & .20 \\ .10 & .30 & .60 \end{bmatrix} \end{array}$$

Determine the steady-state probabilities and the number of customers expected to trade at each bank in the long run.

17. Students switch among the various colleges of a university according to the following probability transition matrix:

This Fall ***Next Fall***

$$\begin{array}{c c} & \begin{array}{ccc} \text{Engineering} & \text{Liberal Arts} & \text{Business} \end{array} \\ \begin{array}{l} \text{Engineering} \\ \text{Liberal Arts} \\ \text{Business} \end{array} & \begin{bmatrix} .50 & .30 & .20 \\ .10 & .70 & .20 \\ .10 & .10 & .80 \end{bmatrix} \end{array}$$

Assume that the number of students in each college of the university at the beginning of the fall quarter is as follows:

Engineering	3,000
Liberal Arts	5,000
Business	2,000

a. Forecast the number of students in each college after the end of the third quarter, based on a four-quarter system.
b. Determine the steady-state conditions for the university.

18. A rental firm in the Southeast serves three states—Virginia, North Carolina, and Maryland. The firm has 700 trucks that are rented on a weekly basis and can be rented in any of the three states. The transition matrix for the movement of rental trucks from state to state is as follows:

Week* n** ***Week* n + *1

$$\begin{array}{c c} & \begin{array}{ccc} \text{Virginia} & \text{North Carolina} & \text{Maryland} \end{array} \\ \begin{array}{l} \text{Virginia} \\ \text{North Carolina} \\ \text{Maryland} \end{array} & \begin{bmatrix} .30 & .50 & .20 \\ .60 & .20 & .20 \\ .40 & .10 & .50 \end{bmatrix} \end{array}$$

Determine the steady-state probabilities and the number of trucks in each state in the long run.

19. The Koher Company manufactures precision machine tools. It has a quality management program for maintaining product quality that relies heavily on statistical process control techniques. A machine center operator takes a sample at the end of every hour to see if the process is within statistical control limits. If the process is not within the preestablished limits, it is out of control; if it is within the limits, the process is in control. If the process remains out of control for 2 hours, it is shut down by the operator. The following transition matrix provides probabilities that a machine center will be in control (C), will be out of control (O), or will shut down (S) in the next hour, given the process status in the current hour:

$$\begin{array}{c c} \textit{This Hour} & \textit{Next Hour} \\ & \begin{array}{ccc} \mathrm{C} & \mathrm{O} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{C} \\ \mathrm{O} \\ \mathrm{S} \end{array} & \begin{bmatrix} .75 & .25 & 0 \\ .85 & .10 & .05 \\ .80 & .05 & .15 \end{bmatrix} \end{array}$$

a. Determine the steady-state probabilities for this transition matrix.
b. For a period of 2,000 hours of operating time (i.e., approximately 1 year), how many hours will a machine center process be in control, out of control, or shut down?

20. In Problem 19 the Koher Company is considering a new operator training program that will alter the probability that a machine center process will be out of control, as follows:

$$\begin{array}{c c} \textit{This Hour} & \textit{Next Hour} \\ & \begin{array}{ccc} \mathrm{C} & \mathrm{O} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{C} \\ \mathrm{O} \\ \mathrm{S} \end{array} & \begin{bmatrix} .85 & .15 & 0 \\ .90 & .05 & .05 \\ .90 & .10 & 0 \end{bmatrix} \end{array}$$

Determine the annual savings in the number of hours a process is out of control or shut down with this new program.

21. Whitesville, where State University is located, has three bookstores that sell textbooks—the State Bookstore, the Eagle Bookstore, and Books n' Things. The State Bookstore is operated by the university. Don Williams, the manager of the State Bookstore, is in the process of placing book orders with his book distributors for the next semester. There are 17,000 undergraduate students at State, and they all purchase their textbooks at one of the three stores. Students often change the bookstore from which they buy from one semester to the next. Don has sampled a group of students and developed the following transition matrix for student movements between the stores:

$$\begin{array}{c c} \textit{This Semester} & \textit{Next Semester} \\ & \begin{array}{ccc} \text{State} & \text{Eagle} & \text{B\&T} \end{array} \\ T = \begin{array}{l} \text{State} \\ \text{Eagle} \\ \text{B\&T} \end{array} & \begin{bmatrix} .42 & .34 & .24 \\ .57 & .25 & .18 \\ .33 & .26 & .41 \end{bmatrix} \end{array}$$

In this semester, if 9,000 students bought their textbooks at the State Bookstore, 5,000 bought their books at the Eagle Bookstore, and 3,000 students bought their books at Books n' Things, how many are likely to buy their books at these stores next semester? How many students will purchase their books from each store in the long-run future?

22. Don Williams, the manager of the State Bookstore from Problem 21, would like to increase his volume of textbook business. He believes that if he reduces textbook prices by 10%, he can increase his

sales volume. Individual student textbook purchases currently average \$175. He is not sure the increase in volume will make up the loss of revenue from the price cut; however, he does think that the additional customers will increase sales for other items, such as clothing, supplies, and computer software. The other stores cannot as easily implement a price cut because they do not carry all the other items the State Bookstore does. Don estimates that a price reduction will alter the transition matrix for the movement of students between stores as follows:

$$\begin{array}{ll} \textbf{\textit{This Semester}} & \textbf{\textit{Next Semester}} \\ & \begin{array}{ccc} \text{State} & \text{Eagle} & \text{B\&T} \end{array} \\ T = \begin{array}{l} \text{State} \\ \text{Eagle} \\ \text{B\&T} \end{array} & \begin{bmatrix} .52 & .29 & .19 \\ .63 & .22 & .15 \\ .39 & .23 & .38 \end{bmatrix} \end{array}$$

However, the other bookstores frequently complain that the State Bookstore has an unfair competitive advantage because it is located on campus and does not charge state sales tax. The university is sensitive to these local business complaints and has indicated to Don that he should keep his total market share in the long run to about 50% or less of the total student body market share.

a. Should Don implement his price reduction strategy?
b. If Don does implement the price cut, how much of an increase or decrease in sales revenue can he expect?

23. At 4:00 P.M. each weekday, the three local television stations in Salem have no network obligations and can schedule whatever shows they choose. WALS runs the Ofrah Williams talk show, WBDJ runs the Josie Donald talk show, and WCXI runs episodes of the Barney Fife show. The transition matrix of probabilities that a regular viewer will watch the same show or change shows from one day to the next is as follows:

$$\begin{array}{ll} \textbf{\textit{Day One}} & \textbf{\textit{Next Day}} \\ & \begin{array}{ccc} \text{Ofrah} & \text{Josie} & \text{Barney} \end{array} \\ T = \begin{array}{l} \text{Ofrah} \\ \text{Josie} \\ \text{Barney} \end{array} & \begin{bmatrix} .67 & .23 & .10 \\ .36 & .58 & .06 \\ .13 & .16 & .71 \end{bmatrix} \end{array}$$

a. Determine the steady-state probabilities for this transition matrix.
b. If there are 27,000 regular viewers in the Salem market at 4:00 P.M., how many can be expected to watch each show in the long run?

24. The Josie Donald show in Problem 23 is contemplating an advertising campaign to increase its viewers, at a cost of \$25,000. Each viewer generates \$0.12 in commercial revenue per day. The station manager knows that the effects of any ad campaign will last only 4 months. The revised transition matrix resulting from the ad campaign is as follows:

$$\begin{array}{ll} \textbf{\textit{Day One}} & \textbf{\textit{Next Day}} \\ & \begin{array}{ccc} \text{Ofrah} & \text{Josie} & \text{Barney} \end{array} \\ T = \begin{array}{l} \text{Ofrah} \\ \text{Josie} \\ \text{Barney} \end{array} & \begin{bmatrix} .58 & .34 & .08 \\ .30 & .65 & .05 \\ .11 & .31 & .58 \end{bmatrix} \end{array}$$

Should the station undertake the ad campaign?

25. Klecko's Copy Center uses several copy machines that deteriorate rather rapidly in terms of the quality of copies produced as the volume of copies increases. Each machine is examined at the end

of each day to determine the quality of the copies being produced, and the results of that inspection are classified as follows:

Classification	Copy Quality	Maintenance Cost per Day
1	Excellent	$ 0
2	Acceptable	100
3	Marginal	400
4	Unacceptable	800

The costs associated with each classification are for maintenance and repair and redoing unacceptable copies. When a machine reaches classification 4 and copies are unacceptable, major maintenance is required (resulting in downtime), after which the machine resumes making excellent copies. The transition matrix showing the probabilities of a machine's being in a particular classification state after inspection is as follows:

$$\begin{array}{cc} \textit{Day One} & \textit{Day Two} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & .8 & .1 & .1 \\ 0 & .6 & .2 & .2 \\ 0 & 0 & .5 & .5 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Determine the expected daily cost of machine maintenance.

26. Determine the steady-state probabilities for the transition matrix in Problem 1.

27. When freshmen at Tech attend orientation, the university's president tells each freshman that one of the two students next to him or her will not graduate. The freshmen interpret this as meaning that two-thirds of the entering freshmen will graduate. The following transition probabilities have been developed from data gathered by Tech's registrar. They show the probability of a student's moving from one class to the next during an academic year and eventually graduating (G) or dropping out (D):

$$\begin{array}{cc} & \begin{array}{cccccc} \text{F} & \text{So} & \text{J} & \text{Sr} & \text{D} & \text{G} \end{array} \\ \begin{array}{c} \text{F} \\ \text{So} \\ \text{J} \\ \text{Sr} \\ \text{D} \\ \text{G} \end{array} & \begin{bmatrix} .10 & .70 & 0 & 0 & .20 & 0 \\ 0 & .10 & .80 & 0 & .10 & 0 \\ 0 & 0 & .15 & .75 & .10 & 0 \\ 0 & 0 & 0 & .15 & .05 & .80 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

a. Is the president's remark during orientation accurate?
b. What is the probability that a freshman will eventually drop out?
c. How many years can an entering freshman expect to remain at Tech?

28. Libby Jackson is a carpenter who works for a large construction company that has a number of housing developments under way in the metropolitan Washington, DC, area. Each day Libby is assigned to one of the company's developments, either hanging and finishing drywall, doing trim

work, framing, or roofing. The following transition matrix describes the probability that she will move from a job one day to the same job or to another the next:

$$T = \begin{array}{c|cccc} \textbf{Day 1} \backslash \textbf{Day 2} & \text{Drywall} & \text{Trim} & \text{Framing} & \text{Roofing} \\ \hline \text{Drywall} & .45 & .23 & .18 & .14 \\ \text{Trim} & .31 & .37 & .23 & .09 \\ \text{Framing} & .28 & .15 & .52 & .05 \\ \text{Roofing} & .26 & .32 & .18 & .24 \end{array}$$

a. Determine the steady-state probabilities for this transition matrix.
b. If Libby works 250 days during the year, how many days will she work at each job?

29. Frank Beamish, the head football coach at Tech, has had his staff scout State University for most of its games this season to get ready for the annual season-ending game. The Tech coaching staff has developed the following transition matrix of the probabilities that State will change its defense from one play to the next:

$$T = \begin{array}{c|cccc} \textbf{This Play} \backslash \textbf{Next Play} & 54 & 63 & \text{Nickel} & \text{Blitz} \\ \hline 54 & .48 & .23 & .19 & .10 \\ 63 & .18 & .55 & .17 & .10 \\ \text{Nickel} & .30 & .40 & .15 & .15 \\ \text{Blitz} & .25 & .50 & .20 & .05 \end{array}$$

a. Determine the steady-state probabilities for this transition matrix.
b. If Tech runs 115 plays during the game, in how many of the plays will State be in each defense?
c. If Tech splits its plays evenly between the pass and run, how many times during the game will State be blitzing when Tech is passing?

Case Problem

The Friendly Car Farm

Buddy Friendly and his wife, Vera, own and manage the Friendly Car Farm, a large new and used car dealership. Buddy and Vera primarily stock family-oriented cars like vans and midsize sedans and pickup trucks. The profit margin on these types of cars is not much, but the Friendlys have a very high volume of business, which offsets their low per-unit profit margin. They feel like they know their customer preferences, and they have tended to stock accordingly. However, for several years one of the automobile manufacturers the Friendlys deal with has repeatedly tried to get them to sell its new, high-priced sports car, the Zephyr AK2000. Buddy and Vera have finally agreed to stock the AK2000, and they have developed a tentative order policy as follows. If the number of Zephyrs they have on the lot is one or fewer at the end of the month, they will order either two or three Zephyrs from the manufacturer so that they will never have more than three of the cars on the lot at the beginning of the month. (The new cars will arrive in the week following the order, at the first of the next month.) The Friendlys paid a statistical consultant from a nearby university to develop the following transition matrix:

$$T = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .08 & .18 & .37 & .37 \\ .63 & .37 & 0 & 0 \\ .26 & .37 & .37 & 0 \\ .08 & .18 & .37 & .37 \end{bmatrix} \end{array}$$

(The consultant used a Poisson distribution with λ = one sale per month to compute each transition probability—a fact completely lost on the Friendlys.)

This transition matrix shows the probability of having either zero, one, two, or three cars in stock one month, given either zero, one, two, or three cars in stock the previous month. For example, if the Friendlys have no Zephyrs in stock in one month, the probability is only .08 that they will have none in stock the next month. The reason this probability is so small is that if the stock level is zero, a new order is placed for three cars. Thus, .08 is actually the probability that three cars were ordered and they were sold in the next month.

The Friendlys pay the manufacturer for cars using loans from a local bank. The interest on these loans is part of the inventory holding cost, which also includes maintenance costs while the car is on the lot. The inventory holding cost for the Zephyr is relatively high compared with that for other cars the Friendlys sell. In any month the holding cost for one Zephyr is $75, for two Zephyrs it is $175, and for three Zephyrs it is $310. (Notice that the holding cost per car increases at an increasing rate because of accumulating interest charges.)

Determine the probability of the Friendlys' having zero, one, two, or three cars in stock in a month in the long run, the expected number of cars in stock in a month in the long-run future, and the average inventory holding cost per month.

Case Problem

Davidson's Department Store

As a result of intense competition and an economic recession, Davidson's Department Store in Atlanta was forced to pay particularly close attention to its cash flow. Because of the poor economy, a number of Davidson's customers were not paying their bills upon receipt, delaying payment for several months, and frequently not paying at all. In general, the Davidson's policy for accounts receivable was to allow a customer to be 2 months late on his or her bill before turning it over to a collection agency. However, it was not quite as simple as that.

Davidson's has approximately 10,000 open accounts at any time. The age of the account is determined by the oldest dollar owed. This means that a customer can have a balance for items bought in two different months, with the overall account being listed as old as the earliest month of purchase. For example, suppose a customer has a balance of $100 at the end of January, $80 of which is for items bought in January and $20 for items bought in November. This means the account is 2 months old at the end of January because the oldest amount on account is from November. If the customer subsequently pays $20 on the bill in February, this cancels the November purchase. Then if the customer makes $100 worth of purchases in February, the account is $180, and it is 1 month old (since the oldest purchases were from January).

Carla Reata, Davidson's comptroller, analyzed the accounts receivable data for the store for an extended period. She summarized these data and developed some probabilities for the payment (or nonpayment) of bills. She determined that for current bills (in their first month of billing), there is an .86 probability that the bills will be paid in the month and a .14 probability that they will be carried over to the next month and be 1 month late. If a bill is already 1 month late, there is a .22 probability that the oldest portion of the bill will be paid so that it will remain 1 month old, a .46 probability that the entire bill will be carried over so that it is 2 months old, and a .32 probability that the bill will be paid in the month. For bills 2 months old, there is a probability of .54 that the oldest portion will be paid so that the bill remains 1 month old, a .16 probability that the next-oldest portion of the bill will be paid so that it remains 2 months old, a .18 probability that the bill will be paid in the month, and a .12 probability that the bill will be listed as a bad debt and turned over to a collection agency. If a bill is paid or listed as a bad debt, it will no longer move to any other billing status.

Under normal circumstances (i.e., not a holiday season), the store averages $1,350,000 in outstanding bills during an average month; $750,000 of this amount is current, $400,000 is 1 month old, and $200,000 is 2 months old. The vice president of finance for the store wants Carla to determine how much of this amount will eventually be paid or end up as bad debts in a typical month. She also wants Carla to tell her if an average cash reserve of $60,000 per month is enough to cover the expected bad debts that will occur each month. Perform this analysis for Carla.