Application 5.4
Defective Eigenvalues and Generalized Eigenvectors

The goal of this application is the solution of the linear systems like

\[ \mathbf{x}' = \mathbf{A} \mathbf{x}, \tag{1} \]

where the coefficient matrix is the exotic 5-by-5 matrix

\[
\mathbf{A} = \begin{bmatrix}
-9 & 11 & -21 & 63 & -252 \\
70 & -69 & 141 & -421 & 1684 \\
-575 & 575 & -1149 & 3451 & -13801 \\
3891 & -3891 & 7782 & -23345 & 93365 \\
1024 & -1024 & 2048 & -6144 & 24572 \\
\end{bmatrix}
\tag{2}
\]

that is generated by the MATLAB command `gallery(5)`. What is so exotic about this particular matrix? Well, enter it in your calculator or computer system of choice, and then use appropriate commands to show that:

- First, the characteristic equation of \( \mathbf{A} \) reduces to \( \lambda^5 = 0 \), so \( \mathbf{A} \) has the single eigenvalue \( \lambda = 0 \) of multiplicity 5.

- Second, there is only a single eigenvector associated with this eigenvalue, which thus has defect 4.

To seek a chain of generalized eigenvectors, show that \( \mathbf{A}^4 \neq 0 \) but \( \mathbf{A}^5 = 0 \) (the 5×5 zero matrix). Hence any nonzero 5-vector \( \mathbf{u}_1 \) satisfies the equation

\[
(\mathbf{A} - \lambda \mathbf{I})^5 \mathbf{u}_1 = \mathbf{A}^5 \mathbf{u}_1 = 0.
\]

Calculate the vectors \( \mathbf{u}_2 = \mathbf{A} \mathbf{u}_1, \mathbf{u}_3 = \mathbf{A} \mathbf{u}_2, \mathbf{u}_4 = \mathbf{A} \mathbf{u}_3, \) and \( \mathbf{u}_5 = \mathbf{A} \mathbf{u}_4 \) in turn. You should find that \( \mathbf{u}_5 \) is nonzero, and is therefore (to within a constant multiple) the unique eigenvector \( \mathbf{v} \) of the matrix \( \mathbf{A} \). But can this eigenvector \( \mathbf{v} \) you find possibly be independent of your original choice of the starting vector \( \mathbf{u}_1 \neq 0 \)? Investigate this question by repeating the process with several different choices of \( \mathbf{u}_1 \).

Finally, having found a length 5 chain \( \{\mathbf{u}_5, \mathbf{u}_4, \mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_1\} \) of generalized eigenvectors based on the (ordinary) eigenvector \( \mathbf{u}_5 \) associated with the single eigenvalue \( \lambda = 0 \) of the matrix \( \mathbf{A} \), write five linearly independent solutions of the 5-dimensional homogeneous linear system \( \mathbf{x}' = \mathbf{A} \mathbf{x} \).
In the sections that follow we illustrate appropriate Maple, Mathematica, and MATLAB techniques to analyze the $4\times4$ matrix

$$A = \begin{bmatrix}
35 & -12 & 4 & 30 \\
22 & -8 & 3 & 19 \\
-10 & 3 & 0 & -9 \\
-27 & 9 & -3 & -23
\end{bmatrix} \quad (3)$$

of Problem 31 in Section 5.4 of the text. You can use any of the other problems there (especially Problems 23–30 and 32) to practice these techniques.

**Using Maple**

First we enter the matrix in (3):

```maple
with(linalg):
A := matrix(4,4, [ 35, -12,  4,  30, 
22,  -8,  3,  19, 
-10,   3,  0,  -9, 
-27,   9, -3, -23 ]):
```

Then we explore its characteristic polynomial, eigenvalues, and eigenvectors:

$$\text{charpoly}(A, \lambda);$$

$$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1$$

(that is, $(\lambda - 1)^4$)

$$\text{eigenvals}(A);$$

$$1, 1, 1, 1$$

$$\text{eigenvects}(A);$$

$$[[1,4,\{[0,1,3,0],[1,0,1,1]\}]]$$

Thus Maple finds only the two independent eigenvectors

$$w1 := \text{matrix}(4,1, [ 0,  1,  3,  0]);$$
$$w2 := \text{matrix}(4,1, [-1,  0,  1,  1]);$$

associated with the multiplicity 4 eigenvalue $\lambda = 1$, which therefore has defect 2. To explore the situation we set up the $4\times4$ identity matrix and the matrix $B = A - \lambda I$: 
Id := diag(1,1,1,1):
L := 1:
B := evalm( A - L*Id):

When we calculate $B^2$ and $B^3$,

$B^2 := evalm(B &* B);
B^3 := evalm(B2 &* B);

we find that $B^2 \neq 0$ but $B^3 = 0$, so there should be a length 3 chain associated with $\lambda = 1$. Choosing

$u_1 := matrix(4,1,[1,0,0,0]);$

we calculate the further generalized eigenvectors

$u_2 := evalm(B &* u1);
u_3 := evalm(B &* u2);$

Thus we have found the length 3 chain $\{u_3, u_2, u_1\}$ based on the (ordinary) eigenvector $u_3$. (To reconcile this result with Maple's eigenvects calculation, you can check that $u_3 + 42w_2 = 7w_1$.) Consequently four linearly independent solutions of the system $x' = Ax$ are given by

$x_1(t) = w_1 e^t,$
$x_2(t) = u_2 e^t,$
$x_3(t) = (u_2 + u_3 t)e^t,$
$x_4(t) = (u_1 + u_2 t + \frac{1}{2} u_3 t^2)e^t.$
Using *Mathematica*

First we enter the matrix in (3):

\[
A = \begin{bmatrix}
35 & -12 & 4 & 30 \\
22 & -8 & 3 & 19 \\
-10 & 3 & 0 & -9 \\
-27 & 9 & -3 & -23
\end{bmatrix};
\]

Then we explore its characteristic polynomial, eigenvalues, and eigenvectors:

\[
\text{CharacteristicPolynomial}[A, r] = 1 - 4r + 6r^2 - 4r^3 + r^4
\]

(that is, \((r-1)^4\))

\[
\text{Eigenvalues}[A] = \{1, 1, 1, 1\}
\]

\[
\text{Eigenvectors}[A] = \{\{-3,-1,0,3\}, \{0,1,3,0\}, \{0,0,0,0\}, \{0,0,0,0\}\}
\]

Thus *Mathematica* finds only the two independent (nonzero) eigenvectors

\[
w_1 = \{-3,-1,0,3\};
w_2 = \{0,1,3,0\};
\]

associated with the multiplicity 4 eigenvalue \(\lambda = 1\), which therefore has defect 2. To explore the situation we set up the \(4 \times 4\) identity matrix and the matrix \(B = A - \lambda I\):

\[
\text{Id} = \text{DiagonalMatrix}[1,1,1,1];
\]
\[
L = 1;
\]
\[
B = A - L*\text{Id};
\]

When we calculate \(B^2\) and \(B^3\),

\[
B^2 = B.B
\]
\[
B^3 = B^2.B
\]

we find that \(B^2 \neq 0\) but \(B^3 = 0\), so there should be a length 3 chain associated with \(\lambda = 1\). Choosing

\[
u_1 = \{\{1\},\{0\},\{0\},\{0\}\}\]
we calculate

\[
\mathbf{u}_2 = \mathbf{B} \cdot \mathbf{u}_1 \\
\{\{34\}, \{22\}, \{-10\}, \{-27\}\}
\]

\[
\mathbf{u}_3 = \mathbf{B} \cdot \mathbf{u}_2 \\
\{\{42\}, \{7\}, \{-21\}, \{-42\}\}
\]

Thus we have found the length 3 chain \{\mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_1\} based on the (ordinary) eigenvector \mathbf{u}_3. (To reconcile this result with Mathematica's Eigenvectors calculation, you can check that \mathbf{u}_3 + 14\mathbf{w}_1 = -7\mathbf{w}_2.) Consequently four linearly independent solutions of the system \(\mathbf{x}' = \mathbf{A} \mathbf{x}\) are given by

\[
x_1(t) = \mathbf{w}_1 e^t, \\
x_2(t) = \mathbf{u}_2 e^t, \\
x_3(t) = (\mathbf{u}_2 + \mathbf{u}_3 t) e^t, \\
x_4(t) = (\mathbf{u}_1 + \mathbf{u}_2 t + \frac{1}{2} \mathbf{u}_3 t^2) e^t.
\]

**Using MATLAB**

First we enter the matrix in (3):

\[
\mathbf{A} = \begin{bmatrix}
35 & -12 & 4 & 30 \\
22 & -8 & 3 & 19 \\
-10 & 3 & 0 & -9 \\
-27 & 9 & -3 & -23 \\
\end{bmatrix};
\]

Then we proceed to explore its characteristic polynomial, eigenvalues, and eigenvectors.

\[
\text{poly(A)} \\
\text{ans} =
\begin{bmatrix}
1.0000 & -4.0000 & 6.0000 & -4.0000 & 1.0000 \\
\end{bmatrix}
\]

These are the coefficients of the characteristic polynomial, which hence is \((\lambda - 1)^4\). Then

\[
[V, D] = \text{eigensys(A)} \\
V = \\
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
-1 & 3 \\
-1 & 0 \\
\end{bmatrix}
\]
\[ D = \\
[1] \\
[1] \\
[1] \\
[1] \]

Thus MATLAB finds only the two independent eigenvectors
\[
\begin{align*}
w1 &= [1 \ 0 \ -1 \ -1]'; \\
w2 &= [0 \ 1 \ 3 \ 0]';
\end{align*}
\]

associated with the single multiplicity 4 eigenvalue \( \lambda = 1 \), which therefore has defect 2. To explore the situation we set up the \( 4 \times 4 \) identity matrix and the matrix \( B = A - \lambda I \):
\[
\begin{align*}
Id &= \text{eye}(4); \\
B &= A - \lambda I; \\
\end{align*}
\]

When we calculate \( B^2 \) and \( B^3 \),
\[
\begin{align*}
B^2 &= B*B \\
B^3 &= B^2*B
\end{align*}
\]

We find that \( B^2 \neq 0 \) but \( B^3 = 0 \), so there should be a length 3 chain associated with the eigenvalue \( \lambda = 1 \). Choosing the first generalized eigenvector
\[
u1 = [1 \ 0 \ 0 \ 0]';
\]

we calculate the further generalized eigenvectors
\[
\begin{align*}
u2 &= B*u1 \\
u2 &= \begin{bmatrix} 34 \\
22 \\
-10 \\
-27 \\
\end{bmatrix}
\end{align*}
\]

and
\[
\begin{align*}
u3 &= B*u2 \\
u3 &= \begin{bmatrix} 42 \\
7 \\
-21 \\
-42 \\
\end{bmatrix}
\end{align*}
\]
Thus we have found the length 3 chain \( \{u_3, u_2, u_1\} \) based on the (ordinary) eigenvector \( u_3 \). (To reconcile this result with MATLAB’s \texttt{eigensys} calculation, you can check that \( u_3 - 42w_1 = 7w_2 \).) Consequently four linearly independent solutions of the system \( x' = Ax \) are given by

\[
\begin{align*}
x_1(t) &= w_1e^t, \\
x_2(t) &= u_3e^t, \\
x_3(t) &= (u_2 + u_1t)e^t, \\
x_4(t) &= (u_1 + u_2t + \frac{1}{2}u_3t^2)e^t.
\end{align*}
\]