Application 10.5B
Rectangular Membrane Vibrations

Here we investigate the vibrations of a flexible membrane whose equilibrium position is the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$. Suppose it is released from rest with given initial displacement, and thereafter its four edges are held fixed. Then (under the usual assumptions) its displacement function $u(x,y,t)$ satisfies the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \left( c^2 = \frac{T}{\rho} \right) \quad (1)$$

$$u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0 \quad (2)$$

$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = 0. \quad (3)$$

According to Problem 3 in Section 10.5 of the text, the solution is given by

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \gamma_{mn}ct \quad (4)$$

where the coefficients are defined by

$$c_{mn} = \frac{4}{ab} \int_{0}^{a} \int_{0}^{b} f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dy \, dx. \quad (5)$$

The $m$th term in (4) corresponds to the membrane's $m$th natural mode of oscillation with displacement function

$$u_{mn}(x,y,t) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \gamma_{mn}ct \quad (6)$$

with circular frequency $\omega_{mn} = \gamma_{mn}c$ where

$$\gamma_{mn}^2 = \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2. \quad (7)$$

The $m$th initial position function

$$u_{mn}(x,y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (8)$$
is the rectangular membrane's \textit{mnth eigenfunction}.

\textbf{Investigation}

For simplicity, take \(a = b = c = 1\) and plot some eigenfunctions with small values of \(m\) and \(n\) in (8). Then plot linear combinations of several eigenfunctions to see some of the more interesting possible initial shapes of a vibrating membrane. For example, the figure below shows the graph of the initial position function

\[
u(x, y) = u_{11}(x, y) - 3 u_{22}(x, y) = \sin x \sin y - 3 \sin 2x \sin 2y
\]

generated by the Mathematica commands

\begin{verbatim}
  u = Sin[x] Sin[y] - 3 Sin[2x] Sin[2y];
  Plot3D[ Evaluate[u], {x,0,Pi}, {y,0,Pi},
           PlotPoints -> {20,20},
           Shading -> False,
           ViewPoint -> {-1.5,3,0.5} ]
\end{verbatim}

The Maple commands

\begin{verbatim}
  u := sin(x)*sin(y) - 3*sin(2*x)*sin(2*y):
  plot3d(u, x=0..Pi, y=0..Pi);
\end{verbatim}

and the MATLAB commands

\begin{verbatim}
  x = 0 : pi/20 : pi;   y = x;
  [x,y] = meshgrid(x,y);
  u = sin(x).*sin(y) - 3*sin(2*x).*sin(2*y);
  surf(x,y,u)
\end{verbatim}

produce similar results.
Maple, Mathematica, and MATLAB all have the capability to animate a sequence of snapshots of a vibrating membrane so as to show a "movie" illustrating its motion. For instance the Maple commands

\[
u := (x,y) \rightarrow \sin(x) \cdot \sin(2y) + \sin(2x) \cdot \sin(y) :
\]

\[w := \sqrt{5} : \quad \text{# circular frequency}\]

\[p := 2\cdot\pi/w : \quad \text{# period of oscillation}\]

with(plots):

\[\text{animate3d}( u(x,y) \cdot \cos(w \cdot t), \]

\[x=0..\pi, y=0..\pi, t=0..p, \]

\[\text{frames=12, style = patch );}\]

produce a 12-frame movie showing one complete oscillation of the membrane with initial position function

\[u(x,y) = \sin x \sin 2y + \sin 2x \sin y \quad \tag{9}\]

and circular frequency \( \omega = \sqrt{5} \). The Mathematica commands

\[u = \sin[x] \cdot \sin[2y] + \sin[2x] \cdot \sin[y];\]

\[w = \sqrt[5]; \quad (* \text{circular frequency } *)\]

\[P = 2 \cdot \pi/w; \quad (* \text{period of oscillation } *)\]

\[\text{frame} = \text{Table}[ \text{Plot3D}[ \text{Evaluate}[u \cdot \cos[w \cdot t]], \]

\[\{x,0,\pi\}, \{y,0,\pi\}, \]

\[\text{PlotRange} \rightarrow \{-1.5, 1.5\}, \]

\[\text{BoxRatios} \rightarrow \{3,3,2\}, \]

\[\text{ViewPoint} \rightarrow \{-1.5, 2.8, 0.75\}, \]

\[\{t,0,P/2, P/20\} ];\]

produce a movie of a half-oscillation which (with the Animate Graphics selection) can be played back-and-forth to show successive oscillations continuously. The command

\[
\text{Show}[ \text{GraphicsArray}[ \{\{\text{frame}[[1]], \text{frame}[[3]]\}, \]

\[\{\text{frame}[[5]], \text{frame}[[7]]\}, \]

\[\{\text{frame}[[9]], \text{frame}[[11]]\}\} ]]\]

displays the array of successive snapshots shown on the next page. Corresponding MATLAB commands are included in the m-file for this project that can be downloaded from the DE projects page at the web site www.prenhall.com/edwards.

Experiment in this way with linear combinations of two, three, or more membrane eigenfunctions of the form

\[u_{mn}(x,y,t) = \sin mx \sin ny \cos \omega_{mn} t \quad \tag{10}\]
where \( \omega_{mn} = \sqrt{m^2 + n^2} \). Vary the coefficients so as to produce a visually attractive movie of sufficient complexity to be interesting.

The Plucked Square Membrane

Suppose the square membrane \( 0 \leq x, y \leq \pi \) is plucked at its center point and set in motion from rest with the initial position function

\[
    u(x, y, 0) = f(x, y) = \min(x, y, \pi - x, \pi - y)
\]

whose graph over the square \( 0 \leq x, y \leq \pi \) looks like a square tent or pyramid with height \( \pi/2 \) at its center. Thus the "tent function" \( f(x, y) \) is the 2-dimensional analogue of the familiar 1-dimensional triangle function that describes the initial position of a plucked string. It can be defined "piecewise" as indicated in the figure on the next page. This diagram indicates how to subdivide the domain of definition of the function \( f \) in the integral in (5) — with \( a = b = \pi \) — in order to calculate the coefficients \( \{c_{mn}\} \) in (4).
Evidently we can write

\[
    f(x, y) = \begin{cases} 
        x & \text{if } x < y < \pi - x, \ 0 < x < \pi / 2, \\
        y & \text{if } y < x < \pi - y, \ 0 < y < \pi / 2, \\
        \pi - y & \text{if } \pi - y < x < y, \ \pi / 2 < y < \pi, \\
        \pi - x & \text{if } \pi - x < y < x, \ \pi / 2 < x < \pi.
    \end{cases}
\]

We proceed here with a Maple-based investigation of the motion of the square membrane if it starts from rest with the initial position function defined in (11). Analogous Mathematica- and MATLAB-based investigations can downloaded from the DE computing projects web site mentioned previously.

The coefficient integral in (5) is the sum of four double integrals corresponding to the four triangles in the figure above. To evaluate these integrals symbolically, we enter the Maple commands

\[
\begin{align*}
    I_1 & := \text{int( int(x*sin(m*x)*sin(n*y),} \\
            y=x..\text{Pi-x}, \ x=0..\text{Pi/2}): \\
    I_2 & := \text{int( int(y*sin(m*x)*sin(n*y),} \\
            x=y..\text{Pi-y}, \ y=0..\text{Pi/2}): \\
    I_3 & := \text{int( int((\text{Pi-y})*sin(m*x)*sin(n*y),} \\
            x=\text{Pi-y}..\text{y}, \ y=\text{Pi/2}..\text{Pi}): \\
    I_4 & := \text{int( int((\text{Pi-x})*sin(m*x)*sin(n*y),} \\
            y=\text{Pi-x}..\text{x}, \ x=\text{Pi/2}..\text{Pi}): \\
\end{align*}
\]
Then the sum in (5) is given by

\[
c := \text{simplify}\left((4/Pi^2)*(I1+I2+I3+I4)\right);
\]

\[
c := 4 \frac{-n \sin(\pi n) + n \cos(\pi m) \sin(\pi m) + \sin(\pi m) \cos(\pi n) m}{m n \pi^2 (-m^2 + n^2)}
\]

We see (from the denominator here) that Maple is assuming \( m \) and \( n \) not equal, in which case it is obvious that \( c_{mn} = 0 \) because of the sine factors.

To calculate the non-zero "diagonal coefficients" in the Fourier series (4), we repeat the computation above with \( m = n \) from the beginning. The result is

\[
c_{nn} = \frac{2\left[1 - (-1)^n\right]}{\pi n^2} = \begin{cases} 4/\pi n^2 & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}
\] (12)

Thus the Fourier series of the tent function \( f(x, y) \) defined in (11) is

\[
f(x, y) = \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin nx \sin ny}{n^2}.
\] (13)

It follows that the solution of our original vibrating membrane problem with initial position function \( f(x, y) \) is given by

\[
u(x, y, t) = \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin nx \sin ny \cos n t \sqrt{2}}{n^2}.
\] (14)

We invite you check out the computation outlined here using either Maple, Mathematica, or MATLAB. Try unequal values of \( a \) and \( b \) to see whether you still get a "diagonal" series as in (14).

Is it now clear — because (14) contains \( n \) terms with \( m \neq n \) — that the function \( u(x, y, t) \) is periodic (in \( t \)) with period \( P = \pi \sqrt{2} \)? This fact, that the tent function in (14) thus yields a "musical" vibration of a square membrane, was first pointed out to us by John Polking of Rice University.

To investigate this vibration visually, we proceed to define a partial sum of the series in (14).
\[ c := n \rightarrow \frac{4}{(\pi n^2)}: \quad \text{# with n odd} \]
\[ N := 13: \quad \text{# so } 2N-1 = 25 \]
\[ P := \pi \sqrt{2}: \quad \text{# period of oscillation} \]
\[ u := (x,y,t) \rightarrow \sum_{k=1}^{N} c(2k-1) \sin((2k-1)x) \sin((2k-1)y) \cos((2k-1)\frac{P t}{\pi}), \quad k=1..N): \]

The following commands plot now snapshots of the resulting vibration with \( t = 0 \) and with \( t = P \).
\[
\begin{align*}
\text{plot3d}(u(x,y,0), \quad x=0..\pi, \quad y=0..\pi); \\
\text{plot3d}(u(x,y,P/8),x=0..\pi, \quad y=0..\pi);
\end{align*}
\]

Thus a vibrating plucked membrane exhibits a "flat spot" that is reminiscent of the flat spot we see in vibrations of a plucked string. Finally, the commands

\[
\text{with(plots):} \\
\text{animate3d( } u(x,y,t), \quad \text{frames}=17, \quad \text{style = patch } );
\]

construct a 17-frame movie that you can play (using the Animate menu) — either continuously or one-frame-at-a-time — to investigate the periodic oscillation. For instance, you will find that the 8th frame (after one-half oscillation) shows a pyramid "pointed" downward instead of upward.