and \( Wx \), and so on. We call these propositional functions \textit{simple predicates}, to distinguish them from more complex propositional functions to be introduced in following sections. A \textit{simple predicate} is a propositional function that has some true and some false substitution instances, each of which is an affirmative singular proposition.

\section*{10.3 Universal and Existential Quantifiers}

A singular proposition affirms that some individual thing has a given predicate, so it is the substitution instance of some propositional function. If the predicate is \( M \) for mortal, or \( B \) for beautiful, we have the simple predicates \( Mx \) or \( Bx \), which assert humanity or beauty of nothing in particular. If we substitute Socrates for the variable \( x \), we get singular propositions, “Socrates is mortal,” or “Socrates is beautiful.” But we might wish to assert that the attribute in question is possessed by more than a single individual. We might wish to say that “Everything is mortal,” or that “Something is beautiful.” These expressions contain predicate terms, but they are not singular propositions because they do not refer specifically to any particular individuals. These are \textit{general} propositions.

Let us look closely at the first of these general propositions, “Everything is mortal.” It may be expressed in various ways that are logically equivalent. We could express it by saying “All things are mortal.” Or we could express it by saying:

\begin{quote}
Given any individual thing whatever, it is mortal.
\end{quote}

In this latter formulation the word “it” is a relative pronoun that refers back to the word “thing” that precedes it. We can use the letter \( x \), our individual variable, in place of both the pronoun and its antecedent. So we can rewrite the first general proposition as

\begin{quote}
Given any \( x \), \( x \) is mortal.
\end{quote}

Or, using the notation for predicates we introduced in the preceding section, we may write

\begin{quote}
Given any \( x \), \( Mx \).
\end{quote}

We know that \( Mx \) is a propositional function, not a proposition. But here, in this last formulation, we have an expression that \textit{contains} \( Mx \), and that clearly \textit{is} a proposition. The phrase “Given any \( x \)” is customarily symbolized by “(\( x \)),” which is called the \textit{universal quantifier}. That first general proposition may now be completely symbolized as

\begin{quote}
(\( x \)) \( Mx \)
\end{quote}

which says, with great penetration, “Everything is mortal.”
This analysis shows that we can convert a propositional function into a proposition not only by substitution, but also by generalization, or quantification. Consider now the second general proposition we had entertained: “Something is beautiful.” This may also be expressed as

There is at least one thing that is beautiful.

In this latter formulation, the word “that” is a relative pronoun referring back to the word “thing.” Using our individual variable \(x\) once again in place of both the pronoun “that” and its antecedent “thing,” we may rewrite the second general proposition as

There is at least one \(x\) such that \(x\) is beautiful.

Or, using the notation for predicates, we may write

There is at least one \(x\) such that \(Bx\).

Once again we see that, although \(Bx\) is a propositional function and not a proposition, we have here an expression that contains \(Bx\) that is a proposition. The phrase “there is at least one \(x\) such that” is customarily symbolized by “\((\exists x)\)” which is called the existential quantifier. Thus the second general proposition may be completely symbolized as

\[(\exists x) Bx\]

which says, with great penetration, “Something is beautiful.”

Thus we see that propositions may be formed from propositional functions either by instantiation, that is, by substituting an individual constant for its individual variable, or by generalization, that is, by placing a universal or existential quantifier before it.

Now consider: The universal quantification of a propositional function, \((x)Mx\), is true if and only if all its substitution instances are true; that is what universality means here. It is also clear that the existential quantification of a propositional function, \((\exists x)Mx\), is true if and only if it has at least one true substitution instance. Let us assume (what no one would deny) that there exists at least one individual. Under this very weak assumption, every propositional function must have at least one substitution instance, an instance that may or may not be true. But it is certain that, under this assumption, if the universal quantification of a propositional function is true, then the existential quantification of it must also be true. That is, if every \(x\) is \(M\), then, if there exists at least one thing, that thing is \(M\).

Up to this point, only affirmative singular propositions have been given as substitution instances of propositional functions. \(Mx\) (\(x\) is mortal) is a propositional function. \(Ms\) is an instance of it, an affirmative singular proposition that says “Socrates is mortal.” But not all propositions are affirmative. One may
deny that Socrates is mortal, saying ~Ms, “Socrates is not mortal.” If Ms is a substitution instance of Mx, then ~Ms may be regarded as a substitution instance of the propositional function ~Mx. And thus we may enlarge our conception of propositional functions, beyond the simple predicates introduced in the preceding section, to permit them to contain the negation symbol, “~.”

With the negation symbol at our disposal, we may now enrich our understanding of quantification as follows. We begin with the general proposition

Nothing is perfect.

which we can paraphrase as

Everything is imperfect.

which in turn may be written as

Given any individual thing whatever, it is not perfect.

which can be rewritten as

Given any x, x is not perfect.

If P symbolizes the attribute of being perfect, we can use the notation just developed (the quantifier and the negation sign) to express this proposition (“Nothing is perfect.”) as (x) ~Px.

Now we are in a position to list and illustrate a series of important connections between universal and existential quantification.

First, the (universal) general proposition “Everything is mortal” is denied by the (existential) general proposition “Something is not mortal.” Using symbols, we may say that (x)Mx is denied by (∃x) ~Mx. Because each of these is the denial of the other, we may certainly say (prefacing the one with a negation symbol) that the biconditional

\[ \sim(x)Mx \iff (\exists x) \sim Mx \]

is necessarily, logically true.

Second, “Everything is mortal” expresses exactly what is expressed by “There is nothing that is not mortal”—which may be formulated as another biconditional, also logically true:

\[ \sim(x)Mx \iff \sim(\exists x) \sim Mx \]

Third, it is clear that the (universal) general proposition, “Nothing is mortal,” is denied by the (existential) general proposition, “Something is mortal.” In symbols we say that (x) ~Mx is denied by (∃x)Mx. And because each of these is the denial of the other, we may certainly say (again prefacing the one with a negation symbol) that the biconditional

\[ \sim(x)\sim Mx \iff (\exists x)Mx \]

is necessarily, logically true.
And fourth, “Everything is not mortal” expresses exactly what is expressed by “There is nothing that is mortal”—which may be formulated as a logically true biconditional:

\[(x) \sim Mx \equiv \sim (\exists x)Mx\]

These four logically true biconditionals set forth the interrelations of universal and existential quantifiers. We may replace any proposition in which the quantifier is prefaced by a negation sign (using these logically true biconditionals) with another logically equivalent proposition in which the quantifier is not prefaced by a negation sign. We list these four biconditionals again, now replacing the illustrative predicate \( M \) (for mortal) with the symbol \( \Phi \) (the Greek letter \( \phi \)), which will stand for any simple predicate whatsoever.

\[
\begin{align*}
[(x) \Phi x] & \equiv [\sim (\exists x) \sim \Phi x] \\
[(\exists x) \Phi x] & \equiv [\sim (x) \sim \Phi x] \\
[(x) \sim \Phi x] & \equiv [\sim (\exists x) \Phi x] \\
[(\exists x) \sim \Phi x] & \equiv [\sim (x) \Phi x]
\end{align*}
\]

Graphically, the general connections between universal and existential quantification can be described in terms of the square array shown in Figure 10-1.
Continuing to assume the existence of at least one individual, we can say, referring to this square, that:

1. The two top propositions are *contraries*; that is, they may both be false but they cannot both be true.
2. The two bottom propositions are *subcontraries*; that is, they may both be true but they cannot both be false.
3. Propositions that are at opposite ends of the diagonals are *contradictories*, of which one must be true and the other must be false.
4. On each side of the square, the truth of the lower proposition is implied by the truth of the proposition directly above it.

### 10.4 Traditional Subject–Predicate Propositions

Using the existential and universal quantifiers, and with an understanding of the square of opposition in Figure 10-1, we are now in a position to analyze (and to use accurately in reasoning) the four types of general propositions that have been traditionally emphasized in the study of logic. The standard illustrations of these four types are the following:

- **All humans are mortal.** (universal affirmative: A)
- **No humans are mortal.** (universal negative: E)
- **Some humans are mortal.** (particular affirmative: I)
- **Some humans are not mortal.** (particular negative: O)

Each of these types is commonly referred to by its letter: the two affirmative propositions, **A** and **I** (from the Latin *affirmo*, I affirm); and the two negative propositions, **E** and **O** (from the Latin *nego*, I deny).*

In symbolizing these propositions by means of quantifiers, we are led to a further enlargement of our conception of a propositional function. Turning first to the **A** proposition, “All humans are mortal,” we proceed by means of successive paraphrasings, beginning with

> Given any individual thing whatever, if it is human then it is mortal.

The two instances of the relative pronoun “it” clearly refer back to their common antecedent, the word “thing.” As in the early part of the preceding

*An account of the traditional analysis of these four types of propositions was presented in Chapter 5.*