10.1 The Need for Quantification

Many valid deductive arguments cannot be tested using the logical techniques of the preceding two chapters. Therefore we must now enhance our analytical tools. We do this with quantification, a twentieth century development chiefly credited to Gottlob Frege (1848–1945), a great German logician and the founder of modern logic. His discovery of quantification has been called the deepest single technical advance ever made in logic.

To understand how quantification increases the power of logical analysis, we must recognize the limitations of the methods we have developed so far. The preceding chapters have shown that we can test deductive arguments effectively—but only arguments of one certain type, those whose validity depends entirely on the ways in which simple statements are truth-functionally combined into compound statements. Applying elementary argument forms and the rule of replacement, we draw inferences that permit us to discriminate valid from invalid arguments of that type. This we have done extensively.

When we confront arguments built of propositions that are not compound, however, those techniques are not adequate; they cannot reach the critical elements in the reasoning process. Consider, for example, the ancient argument

All humans are mortal.
Socrates is human.
Therefore Socrates is mortal.*

*It was to arguments of this type that the classical or Aristotelian logic was primarily devoted, as described in Chapters 5 and 6. Those traditional methods, however, do not possess the generality or power of the newer symbolic logic and cannot be extended to cover all deductive arguments of the kinds we are likely to confront.
This argument is obviously valid. But using the methods so far, introduced we can only symbolize it as

\[ A \]

\[ H \]

\[ \therefore M \]

and on this analysis it appears to be invalid. What is wrong here? The difficulty arises from the fact that the validity of this argument, which is intuitively clear, depends on the inner logical structure of its premises, and that inner structure cannot be revealed by the system of symbolizing statements that we have developed thus far. The symbolization immediately above, plainly too blunt, is the best we can do without quantifiers. That is because the propositions in this valid argument are not compound, and the techniques presented thus far, which are designed to deal with compound statements, cannot deal adequately with noncompound statements. A method is needed with which noncompound statements can be described and symbolized in such a way that their inner logical structure will be revealed. The theory of quantification provides that method.

Quantification enables us to interpret noncompound premises as compound statements, without loss of meaning. With that interpretation we can then use all the elementary argument forms and the rule of replacement (as we have done with compound statements), drawing inferences and proving validity or invalidity—after which the compound conclusion reached may be transformed (again using quantification) back into the noncompound form with which we began. This technique adds very greatly to the power of our analytical machinery.

The methods of deduction developed earlier remain fundamental; quantifiers do not alter the rules of inference in any way. What has gone before may be called the logic of propositions. We now proceed, using some additional symbolization, to apply these rules of inference more widely, in what is called the logic of predicates. The inner structure of propositions, the relations of subjects and predicates, is brought to the surface and made accessible by quantifiers. Introducing this symbolization is the next essential step.

### 10.2 Singular Propositions

We begin with the simplest kind of noncompound statement, illustrated by the second premise of the illustrative argument above, “Socrates is human.” Statements of this kind have traditionally been called singular propositions. An
affirmative singular proposition asserts that a particular individual has some specified attribute. *Socrates* is the subject term in the present example (as ordinary grammar and traditional logic both agree), and *human* is the predicate term. The subject term denotes a particular individual; the predicate term designates some attribute that individual is said to have.

The same subject term, obviously, can occur in different singular propositions. We may assert that “Socrates is mortal,” or “Socrates is fat,” or “Socrates is wise,” or “Socrates is beautiful.” Of these assertions, some are true (the first and the third), and some are false (the second and fourth).* Similarly, the very same predicate term can occur in different singular propositions. The term *human* is a predicate that appears in each of the following: “Aristotle is human,” “Brazil is human,” “Chicago is human,” and “O’Keeffe is human”—of which the first and fourth are true, while the second and third are false.

An “individual” in this symbolism can refer not only to persons, but to any individual thing, such as a country, a book, a city, or anything of which an attribute (such as human or heavy) can be predicated. Attributes do not have to be adjectives (such as “mortal” or “wise”) as in our examples thus far, but can also be nouns (such as “a human”). In grammar the distinction between adjective and noun is important, of course, but in this context it is not significant. We do not need to distinguish between “Socrates is mortal” and “Socrates is a mortal.” Predicates can also be verbs, as in “Aristotle writes,” which can be expressed alternatively as “Aristotle is a writer.” The critical first step is to distinguish between the subject and the predicate terms, between the individuals and the attributes they may be said to have. We next introduce two different kinds of symbols for referring to individuals and to attributes.

To denote individuals we use (following a very widely adopted convention) small, or lowercase letters, from *a* through *w*. These symbols are individual constants. In any particular context in which they may occur, each will designate one particular individual throughout the whole of that context. It is usually convenient to denote an individual by the first letter of its (or his or her) name. We may use the letter *s* to denote Socrates, *a* to denote Aristotle, *b* to denote Brazil, *c* to denote Chicago, and so forth.

*Here we follow the custom of ignoring the time factor, and use the verb “is” in the tenseless sense of “is, will be, or has been.” Where considerations of time change are crucial, the somewhat more complicated symbolism of the logic of relations is required for an adequate treatment.*
We use capital letters to symbolize attributes that individuals may have, and again it is convenient to use the first letter of the attribute referred to: $H$ for human, $M$ for mortal, $F$ for fat, $W$ for wise, and so forth.

We can now symbolize a singular proposition. By writing an attribute symbol immediately to the left of an individual symbol, we symbolize the singular proposition affirming that the individual named has the attribute specified. Thus the singular proposition, “Socrates is human,” will be symbolized simply as $Hs$. And, of course, $Ha$ symbolizes “Aristotle is human,” $Hb$ symbolizes “Brazil is human,” $Hc$ symbolizes “Chicago is human,” and so forth.

It is important to note the pattern that is common to these terms. Each begins with the same attribute symbol, $H$, and is followed by a symbol for some individual, $s$ or $a$ or $b$ or $c$, and so forth. We could write the pattern as “$H_—$” where the dash to the right of the predicate symbol is a place marker for some individual symbol. This pattern we symbolize as $Hx$. We use $Hx$ [sometimes written as $H(x)$] to symbolize the common pattern of all singular propositions that attribute “being human” to some individual. The letter $x$ is called an individual variable—it is simply a place marker, indicating where the various individual letters $a$ through $w$ (the individual constants) may be written. When one of those constants does appear in place of $x$, we have a singular proposition. The letter $x$ is available to serve as the variable because, by convention, $a$ through $w$ are the only letters we allow to serve as individual constants.

Let us examine the symbol $Hx$ more closely. It is called a propositional function. We define a propositional function as an expression that (1) contains an individual variable and (2) becomes a statement when an individual constant is substituted for the individual variable.* So a propositional function is not itself a proposition, although it can become one by substitution. Individual constants may be thought of as the proper names of individuals. Any singular proposition is a substitution instance of a propositional function; it is the result of substituting some individual constant for the individual variable in that propositional function.

A propositional function normally has some true substitution instances and some false substitution instances. If $H$ symbolizes human, $s$ symbolizes Socrates, and $c$ symbolizes Chicago, then $Hs$ is true and $Hc$ is false. With the substitution made, what confronts us is a proposition; before the substitution is made, we have only the propositional function. There are an unlimited number of such propositional functions, of course: $Hx$, and $Mx$, and $Bx$, and $Fx$.

*Some writers regard “propositional functions” as the meanings of such expressions, but here we define them to be the expressions themselves.
and \( Wx \), and so on. We call these propositional functions *simple predicates*, to distinguish them from more complex propositional functions to be introduced in following sections. A *simple predicate* is a propositional function that has some true and some false substitution instances, each of which is an affirmative singular proposition.

### 10.3 Universal and Existential Quantifiers

A singular proposition affirms that some individual thing has a given predicate, so it is the substitution instance of some propositional function. If the predicate is \( M \) for mortal, or \( B \) for beautiful, we have the simple predicates \( Mx \) or \( Bx \), which assert humanity or beauty of nothing in particular. If we substitute Socrates for the variable \( x \), we get singular propositions, “Socrates is mortal,” or “Socrates is beautiful.” But we might wish to assert that the attribute in question is possessed by more than a single individual. We might wish to say that “Everything is mortal,” or that “Something is beautiful.” These expressions contain predicate terms, but they are not singular propositions because they do not refer specifically to any particular individuals. These are *general* propositions.

Let us look closely at the first of these general propositions, “Everything is mortal.” It may be expressed in various ways that are logically equivalent. We could express it by saying “All things are mortal.” Or we could express it by saying:

Given any individual thing whatever, it is mortal.

In this latter formulation the word “it” is a relative pronoun that refers back to the word “thing” that precedes it. We can use the letter \( x \), our individual variable, in place of both the pronoun and its antecedent. So we can rewrite the first general proposition as

Given any \( x \), \( x \) is mortal.

Or, using the notation for predicates we introduced in the preceding section, we may write

Given any \( x \), \( Mx \).

We know that \( Mx \) is a propositional function, not a proposition. But here, in this last formulation, we have an expression that contains \( Mx \), and that clearly is a proposition. The phrase “Given any \( x \)” is customarily symbolized by “(\( x \))”, which is called the universal quantifier. That first general proposition may now be completely symbolized as

\[
(x) \ Mx
\]

which says, with great penetration, “Everything is mortal.”
This analysis shows that we can convert a propositional function into a proposition not only by substitution, but also by generalization, or quantification.

Consider now the second general proposition we had entertained: “Something is beautiful.” This may also be expressed as

There is at least one thing that is beautiful.

In this latter formulation, the word “that” is a relative pronoun referring back to the word “thing.” Using our individual variable \( x \) once again in place of both the pronoun “that” and its antecedent “thing,” we may rewrite the second general proposition as

There is at least one \( x \) such that \( x \) is beautiful.

Or, using the notation for predicates, we may write

There is at least one \( x \) such that \( Bx \).

Once again we see that, although \( Bx \) is a propositional function and not a proposition, we have here an expression that contains \( Bx \) that is a proposition. The phrase “there is at least one \( x \) such that” is customarily symbolized by “\((\exists x)\)” which is called the existential quantifier. Thus the second general proposition may be completely symbolized as

\( (\exists x) Bx \)

which says, with great penetration, “Something is beautiful.”

Thus we see that propositions may be formed from propositional functions either by instantiation, that is, by substituting an individual constant for its individual variable, or by generalization, that is, by placing a universal or existential quantifier before it.

Now consider: The universal quantification of a propositional function, \((x)Mx\), is true if and only if all its substitution instances are true; that is what universality means here. It is also clear that the existential quantification of a propositional function, \((\exists x)Mx\), is true if and only if it has at least one true substitution instance. Let us assume (what no one would deny) that there exists at least one individual. Under this very weak assumption, every propositional function must have at least one substitution instance, an instance that may or may not be true. But it is certain that, under this assumption, if the universal quantification of a propositional function is true, then the existential quantification of it must also be true. That is, if every \( x \) is \( M \), then, if there exists at least one thing, that thing is \( M \).

Up to this point, only affirmative singular propositions have been given as substitution instances of propositional functions. \( Mx \) (\( x \) is mortal) is a propositional function. \( Ms \) is an instance of it, an affirmative singular proposition that says “Socrates is mortal.” But not all propositions are affirmative. One may
deny that Socrates is mortal, saying \( \sim M_s \), “Socrates is not mortal.” If \( M_s \) is a substitution instance of \( M_x \), then \( \sim M_s \) may be regarded as a substitution instance of the propositional function \( \sim M_x \). And thus we may enlarge our conception of propositional functions, beyond the simple predicates introduced in the preceding section, to permit them to contain the negation symbol, “\( \sim \).”

With the negation symbol at our disposal, we may now enrich our understanding of quantification as follows. We begin with the general proposition

Nothing is perfect.

which we can paraphrase as

Everything is imperfect.

which in turn may be written as

Given any individual thing whatever, it is not perfect.

which can be rewritten as

Given any \( x \), \( x \) is not perfect.

If \( P \) symbolizes the attribute of being perfect, we can use the notation just developed (the quantifier and the negation sign) to express this proposition (“Nothing is perfect.”) as \( \sim P_x \).

Now we are in a position to list and illustrate a series of important connections between universal and existential quantification.

First, the (universal) general proposition “Everything is mortal” is denied by the (existential) general proposition “Something is not mortal.” Using symbols, we may say that \( (x)M_x \) is denied by \( (\exists x) \sim M_x \). Because each of these is the denial of the other, we may certainly say (prefacing the one with a negation symbol) that the biconditional

\[
\sim (x)M_x \equiv (\exists x) \sim M_x
\]

is necessarily, logically true.

Second, “Everything is mortal” expresses exactly what is expressed by “There is nothing that is not mortal”—which may be formulated as another biconditional, also logically true:

\[
\sim (x)M_x \equiv (\sim (\exists x) \sim M_x)
\]

Third, it is clear that the (universal) general proposition, “Nothing is mortal,” is denied by the (existential) general proposition, “Something is mortal.” In symbols we say that \( (x) \sim M_x \) is denied by \( (\exists x)M_x \). And because each of these is the denial of the other, we may certainly say (again prefacing the one with a negation symbol) that the biconditional

\[
\sim (x) \sim M_x \equiv (\exists x)M_x
\]

is necessarily, logically true.
And fourth, “Everything is not mortal” expresses exactly what is expressed by “There is nothing that is mortal”—which may be formulated as a logically true biconditional:

\[(x)\sim Mx \equiv \sim (\exists x)Mx\]

These four logically true biconditionals set forth the interrelations of universal and existential quantifiers. We may replace any proposition in which the quantifier is prefaced by a negation sign (using these logically true biconditionals) with another logically equivalent proposition in which the quantifier is not prefaced by a negation sign. We list these four biconditionals again, now replacing the illustrative predicate \(M\) (for mortal) with the symbol \(\phi\) (the Greek letter \(phi\)), which will stand for any simple predicate whatsoever.

\[
\begin{align*}
[(x)\phi x] & \equiv [\sim (\exists x)\sim \phi x] \\
[\exists x)\phi x] & \equiv [\sim (x)\sim \phi x] \\
[(x)\sim \phi x] & \equiv [\sim (\exists x)\phi x] \\
[\exists x)\sim \phi x] & \equiv [\sim (x)\phi x]
\end{align*}
\]

Graphically, the general connections between universal and existential quantification can be described in terms of the square array shown in Figure 10-1.
Continuing to assume the existence of at least one individual, we can say, referring to this square, that:

1. The two top propositions are *contraries*; that is, they may both be false but they cannot both be true.
2. The two bottom propositions are *subcontraries*; that is, they may both be true but they cannot both be false.
3. Propositions that are at opposite ends of the diagonals are *contradictories*, of which one must be true and the other must be false.
4. On each side of the square, the truth of the lower proposition is implied by the truth of the proposition directly above it.

### 10.4 Traditional Subject–Predicate Propositions

Using the existential and universal quantifiers, and with an understanding of the square of opposition in Figure 10-1, we are now in a position to analyze (and to use accurately in reasoning) the four types of general propositions that have been traditionally emphasized in the study of logic. The standard illustrations of these four types are the following:

- **All humans are mortal.** (universal affirmative: *A*)
- **No humans are mortal.** (universal negative: *E*)
- **Some humans are mortal.** (particular affirmative: *I*)
- **Some humans are not mortal.** (particular negative: *O*)

Each of these types is commonly referred to by its letter: the two affirmative propositions, *A* and *I* (from the Latin *affirmo*, I affirm); and the two negative propositions, *E* and *O* (from the Latin *nego*, I deny).*

In symbolizing these propositions by means of quantifiers, we are led to a further enlargement of our conception of a propositional function. Turning first to the *A* proposition, “All humans are mortal,” we proceed by means of successive paraphrasings, beginning with

Given any individual thing whatever, if it is human then it is mortal.

The two instances of the relative pronoun “it” clearly refer back to their common antecedent, the word “thing.” As in the early part of the preceding

*An account of the traditional analysis of these four types of propositions was presented in Chapter 5.*
section, because those three words have the same (indefinite) reference, they can be replaced by the letter $x$ and the proposition rewritten as

\[ \text{Given any } x, \text{ if } x \text{ is human then } x \text{ is mortal.} \]

Now using our previously introduced notation for “if–then,” we can rewrite the preceding as

\[ \text{Given any } x, x \text{ is human } \supset x \text{ is mortal.} \]

Finally, using our now-familiar notation for propositional functions and quantifiers, the original $A$ proposition is expressed as

\[ (x)(Hx \supset Mx) \]

In our symbolic translation, the $A$ proposition appears as the universal quantification of a new kind of propositional function. The expression $Hx \supset Mx$ is a propositional function that has as its substitution instances neither affirmative nor negative singular propositions, but conditional statements whose antecedents and consequents are singular propositions that have the same subject term. Among the substitution instances of the propositional function $Hx \supset Mx$ are the conditional statements $Ha \supset Ma$, $Hb \supset Mb$, $Hc \supset Mc$, $Hd \supset Md$, and so on.

There are also propositional functions whose substitution instances are conjunctions of singular propositions that have the same subject terms. Thus the conjunctions $Ha \land Ma$, $Hb \land Mb$, $Hc \land Mc$, $Hd \land Md$, and so on, are substitution instances of the propositional function $Hx \land Mx$. There are also propositional functions such as $Wx \lor Bx$, whose substitution instances are disjunctions such as $Wa \lor Ba$ and $Wb \lor Bb$. In fact, any truth-functionally compound statement whose simple component statements are singular propositions that all have the same subject term may be regarded as a substitution instance of a propositional function containing some or all of the various truth-functional connectives: dot, wedge, horseshoe, three-bar equivalence, and curl, in addition to the simple predicates $Ax$, $Bx$, $Cx$, $Dx$, . . . . In our translation of the $A$ proposition as $(x)(Hx \supset Mx)$, the parentheses serve as punctuation marks. They indicate that the universal quantifier $(x)$ “applies to” or “has within its scope” the entire (complex) propositional function $Hx \supset Mx$.

Before going on to discuss the other traditional forms of categorical propositions, it should be observed that our symbolic formula $(x)(Hx \supset Mx)$ translates not only the standard-form proposition, “All $H$’s are $M$’s,” but any other English sentence that has the same meaning. When, for example, a character in Henrik Ibsen’s play, Love’s Comedy, says, “A friend married is a friend lost,” that is just another way of saying, “All friends who marry are friends who are lost.” There are many ways, in English, of saying the same thing.
Here is a list, not exhaustive, of different ways in which we commonly express universal affirmative propositions in English:

- H’s are M’s.
- An H is an M.
- Every H is M.
- Each H is M.
- Any H is M.
- No H’s are not M.
- Everything that is H is M.
- Anything that is H is M.
- If anything is H, it is M.
- If something is H, it is M.
- Whatever is H is M.
- H’s are all M’s.
- Only M’s are H’s.
- None but M’s are H’s.
- Nothing is an H unless it is an M.
- Nothing is an H but not an M.

To evaluate an argument we must understand the language in which the propositions of that argument are expressed. Some English idioms are a little misleading, using a temporal term when no reference to time is intended. Thus the proposition, “H’s are always M’s,” is ordinarily understood to mean simply that all H’s are M’s. Again, the same meaning may be expressed by the use of abstract nouns: “Humanity implies (or entails) mortality” is correctly symbolized as an A proposition. That the language of symbolic logic has a single expression for the common meaning of a considerable number of English sentences may be regarded as an advantage of symbolic logic over English for cognitive or informative purposes—although admittedly a disadvantage from the point of view of rhetorical power or poetic expressiveness.

### Quantification of the A Proposition

The A proposition, “All humans are mortal,” asserts that if anything is a human, then it is mortal. In other words, for any given thing x, if x is a human, then x is mortal. Substituting the horseshoe symbol for “if–then,” we get

\[
(x) [Hx \rightarrow Mx]
\]
Quantification of the E Proposition

The E proposition, “No humans are mortals,” asserts that if anything is human, then it is not mortal. In other words, for any given thing x, if x is a human, then x is not mortal. Substituting the horseshoe symbol for “if–then,” we get:

Given any x, x is a human ⊃ x is not mortal.

In the notation for propositional functions and quantifiers, this becomes

\( (x) [Hx \supset \neg Mx] \)

This symbolic translation expresses not only the traditional E form in English, but also such diverse ways of saying the same thing as “There are no H's that are M,” “Nothing is both an H and an M,” and “H's are never M.”

Quantification of the I Proposition

The I proposition, “Some humans are mortal,” asserts that there is at least one thing that is a human and is mortal. In other words, there is at least one x such that x is a human and x is mortal. Substituting the dot symbol for conjunction, we get

There is at least one x such that x is a human • x is mortal.

In the notation for propositional functions and quantifiers, this becomes

\( (\exists x) [Hx \cdot Mx] \)

Quantification of the O Proposition

The O proposition, “Some humans are not mortal,” asserts that there is at least one thing that is a human and is not mortal. In other words, there is at least one x such that x is human and x is not mortal. Substituting the dot symbol for conjunction we get

There is a least one x such that x is a human • x is not mortal.

In the notation for propositional functions and quantifiers, this becomes

\( (\exists x) [Hx \cdot \neg Mx] \)

Where the Greek letters phi (Φ) and psi (Ψ) are used to represent any predicates whatever, the four general subject–predicate propositions of traditional logic may be represented in a square array as shown in Figure 10-2.
Of these four, the A and the O are contradictories, each being the denial of the other; E and I are also contradictories.

It might be thought that an I proposition follows from its corresponding A proposition, and an O from its corresponding E, but this is not so. An A proposition may very well be true while its corresponding I proposition is false. Where \( \Phi x \) is a propositional function that has no true substitution instances, then no matter what kinds of substitution instances the propositional function \( \Psi x \) may have, the universal quantification of the (complex) propositional function \( \Phi x \supset \Psi x \) will be true. For example, consider the propositional function, “\( x \) is a centaur,” which we abbreviate as \( Cx \). Because there are no centaurs, every substitution instance of \( Cx \) is false, that is, \( Ca, Cb, Cc, \ldots \), are all false. Hence every substitution instance of the complex propositional function \( Cx \supset Bx \) will be a conditional statement whose antecedent is false. The substitution instances \( Ca \supset Ba, Cb \supset Bb, Cc \supset Bc, \ldots \), are therefore all true, because any conditional statement asserting a material implication must be true if its antecedent is false. Because all its substitution instances are true, the universal quantification of the propositional function \( Cx \supset Bx \), which is the A proposition \((x)(Cx \supset Bx)\), is true. But the corresponding I proposition \((\exists x)(Cx \cdot Bx)\) is false, because the propositional function \( Cx \cdot Bx \) has no true substitution instances. That \( Cx \cdot Bx \) has no true substitution instances follows from the fact that \( Cx \) has no true substitution instances. The various substitution instances of \( Cx \cdot Bx \) are \( Ca \cdot Ba, Cb \cdot Bb, Cc \cdot Bc, \ldots \), each of which is a conjunction whose first conjunct is false, because \( Ca, Cb, Cc, \ldots \), are all false. Because all its substitution instances are false, the existential quantification of the propositional function \( Cx \cdot Bx \), which is the I proposition \((\exists x)(Cx \cdot Bx)\), is false. Hence an A proposition may be true while its corresponding I proposition is false.
This analysis shows also why an E proposition may be true while its corresponding O proposition is false. If we replace the propositional function Bx by the propositional function ~Bx in the preceding discussion, then \((x)(Cx \supset \sim Bx)\) may be true while \((\exists x)(Cx \bullet \sim Bx)\) will be false because, of course, there are no centaurs.

The key to the matter is this: A propositions and E propositions do not assert or suppose that anything exists; they assert only that (if one thing then another) is the case. But I propositions and O propositions do suppose that some things exist; they assert that (this and the other) is the case. The existential quantifier in I and O propositions makes a critical difference. It would plainly be a mistake to infer the existence of anything from a proposition that does not assert or suppose the existence of anything.

If we make the general assumption that there exists at least one individual, then \((x)(Cx \supset Bx)\) does imply \((\exists x)(Cx \supset Bx)\). But the latter is not an I proposition. The I proposition, “Some centaurs are beautiful,” is symbolized as \((\exists x)(Cx \bullet Bx)\), which says that there is at least one centaur that is beautiful. But what is symbolized as \((\exists x)(Cx \supset Bx)\) can be rendered in English as “There is at least one thing such that, if it is a centaur, then it is beautiful.” It does not say that there is a centaur, but only that there is an individual that is either not a centaur or is beautiful. This proposition would be false in only two possible cases: first, if there were no individuals at all; and second, if all individuals were centaurs and none of them was beautiful. We rule out the first case by making the explicit (and obviously true) assumption that there is at least one individual in the universe. And the second case is so extremely implausible that any proposition of the form \((\exists x)(\Phi x \supset \Psi x)\) is bound to be quite trivial, in contrast to the significant I form \((\exists x)(\Phi x \bullet \Psi x)\). The foregoing should make clear that, although in English the A and I propositions “All humans are mortal” and “Some humans are mortal” differ only in their initial words, “all” and “some,” their difference in meaning is not confined to the matter of universal versus existential quantification, but goes deeper than that. The propositional functions quantified to yield A and I propositions are not just differently quantified; they are different propositional functions, one containing “\(\supset\),” the other “\(\bullet\).” In other words, A and I propositions are not as much alike as they appear in English. Their differences are brought out very clearly in the notation of propositional functions and quantifiers.

For purposes of logical manipulation we can work best with formulas in which the negation sign, if one appears at all, applies only to simple predicates. So we will want to replace formulas in ways that have this result. This we can do quite readily. We know from the rule of replacement established in Chapter 9 that we are always entitled to replace an expression by another that is logically equivalent to it; and we have at our disposal four logical
equivalences (listed in Section 10.3) in which each of the propositions in
which the quantifier is negated is shown equivalent to another proposition
in which the negation sign applies directly to the predicates. Using the rules
of inference with which we have long been familiar, we can shift negation
signs so that, in the end, they no longer apply to compound expressions but
apply only to simple predicates. Thus, for example, the formula

$$\neg(\exists x)(Fx \cdot \neg Gx)$$

can be successively rewritten. First, when we apply the third logical equiva-
lence given on page 444, it is transformed into

$$(x) \neg(Fx \cdot \neg Gx)$$

Then when we apply De Morgan’s theorem, it becomes

$$(x)(\neg Fx \lor \neg \neg Gx)$$

Next, the principle of Double Negation gives us

$$(x)(\neg Fx \lor Gx)$$

And finally, when we invoke the definition of Material Implication, the original
formula is rewritten as the A proposition

$$(x)(Fx \supset Gx)$$

We call a formula in which negation signs apply only to simple predicates a
**normal-form formula.**

Before turning to the topic of inferences involving noncompound state-
ments, the reader should acquire some practice in translating noncompound
statements from English into logical symbolism. The English language has so
many irregular and idiomatic constructions that there can be no simple rules
for translating an English sentence into logical notation. What is required in
each case is that the meaning of the sentence be understood and then restated
in terms of propositional functions and quantifiers.

**EXERCISES**

A. Translate each of the following into the logical notation of propositional
functions and quantifiers, in each case using the abbreviations suggested and
making each formula begin with a quantifier, not with a negation symbol.

**EXAMPLE**

1. No beast is without some touch of pity. ($Bx$: $x$ is a beast; $Px$: $x$ has some
touch of pity.)
(x)(Bx ⊃ Px)

2. Sparrows are not mammals. (Sx: x is a sparrow; Mx: x is a mammal.)

3. Reporters are present. (Rx: x is a reporter; Px: x is present.)

4. Nurses are always considerate. (Nx: x is a nurse; Cx: x is considerate.)

5. Diplomats are not always rich. (Dx: x is a diplomat; Rx: x is rich.)

6. "To swim is to be a penguin." (Sx: x swims; Px: x is a penguin.)
   —Christina Slagar, curator, Monterey Bay Aquarium, 17 January 2003

7. No boy scout ever cheats. (Bx: x is a boy scout; Cx: x cheats.)

8. Only licensed physicians can charge for medical treatment. (Lx: x is a licensed physician; Cx: x can charge for medical treatment.)

9. Snake bites are sometimes fatal. (Sx: x is a snake bite; Fx: x is fatal.)

10. The common cold is never fatal. (Cx: x is a common cold; Fx: x is fatal.)

11. A child pointed his finger at the emperor. (Cx: x is a child; Px: x pointed his finger at the emperor.)

12. Not all children pointed their fingers at the emperor. (Cx: x is a child; Px: x pointed his finger at the emperor.)

13. All that glitters is not gold. (Gx: x glitters; Ax: x is gold.)

14. None but the brave deserve the fair. (Bx: x is brave; Dx: x deserves the fair.)

15. Only citizens of the United States can vote in U.S. elections. (Cx: x is a citizen of the United States; Vx: x can vote in U.S. elections.)

16. Citizens of the United States can vote only in U.S. elections. (Ex: x is an election in which citizens of the United States can vote; Ux: x is a U.S. election.)

17. Not every applicant was hired. (Ax: x is an applicant; Hx: x was hired.)

18. Not any applicant was hired. (Ax: x is an applicant; Hx: x was hired.)

19. Nothing of importance was said. (Lx: x is of importance; Sx: x was said.)

20. They have the right to criticize who have a heart to help. (Cx: x has the right to criticize; Hx: x has a heart to help.)
B. Translate each of the following into the logical notation of propositional functions and quantifiers, in each case making the formula begin with a quantifier, not with a negation symbol.

1. Nothing is attained in war except by calculation.  
   —Napoleon Bonaparte

2. No one doesn’t believe in laws of nature.  

3. He only earns his freedom and existence who daily conquers them anew.  
   —Johann Wolfgang Von Goethe, *Faust*, Part II

4. No man is thoroughly miserable unless he be condemned to live in Ireland.  
   —Jonathan Swift

*5. Not everything good is safe, and not everything dangerous is bad.  

6. There isn’t any business we can’t improve.  
   —Advertising slogan, Ernst and Young, Accountants

7. A problem well stated is a problem half solved.  
   —Charles Kettering, former research director for General Motors

8. There’s not a single witch or wizard who went bad who wasn’t in Slytherin.  
   —J. K. Rowling, in *Harry Potter and the Sorcerer’s Stone*

9. Everybody doesn’t like something, but nobody doesn’t like Willie Nelson.  
   —Steve Dollar, Cox News Service

*10. No man but a blockhead ever wrote except for money.  
   —Samuel Johnson

C. For each of the following, find a normal-form formula that is logically equivalent to the given one.

*1. \( \neg (x)(Ax \supset Bx) \)  
2. \( \neg (x)(Cx \supset \neg Dx) \)

3. \( \neg (\exists x)(Ex \cdot Fx) \)  
4. \( \neg (\exists x)(Gx \cdot \neg Hx) \)

*5. \( \neg (x)(\neg Ix \lor Jx) \)  
6. \( \neg (x)(\neg Kx \lor \neg Lx) \)

7. \( \neg (\exists x)[\neg (Mx \lor Nx)] \)  
8. \( \neg (\exists x)[\neg (Ox \lor \neg Px)] \)

9. \( \neg (\exists x)[\neg (\neg Qx \lor Rx)] \)  
*10. \( \neg (x)[\neg (Sx \cdot \neg Tx)] \)

11. \( \neg (x)[\neg (\neg Ux \cdot \neg Vx)] \)  
12. \( \neg (\exists x)[\neg (\neg Wx \lor Xx)] \)
10.5 Proving Validity

In order to construct formal proofs of validity for arguments whose validity turns on the inner structures of noncompound statements that occur in them, we must expand our list of rules of inference. Only four additional rules are required, and they will be introduced in connection with arguments for which they are needed.

Consider the first argument we discussed in this chapter: “All humans are mortal. Socrates is human. Therefore Socrates is mortal.” It is symbolized as

\[(\forall x)(Hx \supset Mx)\]
\[Hs\]
\[\therefore Ms\]

The first premise affirms the truth of the universal quantification of the propositional function \(Hx \supset Mx\). Because the universal quantification of a propositional function is true if and only if all of its substitution instances are true, from the first premise we can infer any desired substitution instance of the propositional function \(Hx \supset Mx\). In particular, we can infer the substitution instance \(Hs \supset Ms\).

From that and the second premise \(Hs\), the conclusion \(Ms\) follows directly by modus ponens.

If we add to our list of rules of inference the principle that any substitution instance of a propositional function can validly be inferred from its universal quantification, then we can give a formal proof of the validity of the given argument by reference to the expanded list of elementary valid argument forms. This new rule of inference is the principle of Universal Instantiation* and is abbreviated as U.I. Using the Greek letter \(\nu\) to represent any individual symbol whatever, we state the new rule as

\[\text{UI: } (\forall x)(\Phi x) \quad \therefore \Phi \nu \quad \text{(where } \nu \text{ is any individual symbol)}\]

A formal proof of validity may now be written as

1. \((\forall x)(Hx \supset Mx)\)
2. \(Hs\)
   \[\therefore Ms\]
3. \(Hs \supset Ms\) \hspace{1em} 1, U.I.
4. \(Ms\) \hspace{1em} 3,2, M.P.

*This rule, and the three following, are variants of rules for natural deduction that were devised independently by Gerhard Gentzen and Stanislaw Jaskowski in 1934.
The addition of U.I. strengthens our proof apparatus considerably, but more is required. The need for additional rules governing quantification arises in connection with arguments such as “All humans are mortal. All Greeks are human. Therefore all Greeks are mortal.” The symbolic translation of this argument is

\((x)(Hx \supset Mx)\)
\((x)(Gx \supset Hx)\)
\(\therefore (x)(Gx \supset Mx)\)

Here both premises and conclusion are general propositions rather than singular ones, universal quantifications of propositional functions rather than substitution instances of them. From the two premises, by U.I., we may validly infer the following pairs of conditional statements:

\[
\begin{align*}
&\{ Ga \supset Ha \} & \{ Gb \supset Hb \} & \{ Gc \supset Hc \} & \{ Gd \supset Hd \} \\
&\{ Ha \supset Ma \} & \{ Hb \supset Mb \} & \{ Hc \supset Mc \} & \{ Hd \supset Md \} \ldots \\
\end{align*}
\]

and by successive uses of the principle of the hypothetical syllogism we may validly infer the conclusions:

\(Ga \supset Ma, Gb \supset Mb, Gc \supset Mc, Gd \supset Md, \ldots\)

If \(a, b, c, d, \ldots\) were all the individuals that exist, it would follow that from the truth of the premises one could validly infer the truth of all substitution instances of the propositional function \(Gx \supset Mx\). The universal quantification of a propositional function is true if and only if all its substitution instances are true, so we can go on to infer the truth of \((x)(Gx \supset Mx)\), which is the conclusion of the given argument.

The preceding paragraph may be thought of as containing an informal proof of the validity of the given argument, in which the principle of the hypothetical syllogism and two principles governing quantification are appealed to. But it describes indefinitely long sequences of statements: the lists of all substitution instances of the two propositional functions quantified universally in the premises, and the list of all substitution instances of the propositional function whose universal quantification is the conclusion. A formal proof cannot contain such indefinitely, perhaps even infinitely, long sequences of statements, so some method must be sought for expressing those indefinitely long sequences in some finite, definite fashion.

A method for doing this is suggested by a common technique of elementary mathematics. A geometer, seeking to prove that all triangles possess a certain attribute, may begin with the words “Let \(ABC\) be any arbitrarily selected triangle.” Then the geometer begins to reason about the triangle \(ABC\), and establishes that it has the attribute in question. From this she concludes that all triangles have that attribute. Now what justifies her final
conclusion? Granted of the particular triangle $ABC$ that it has the attribute, why does it follow that all triangles do? The answer to this question is easily given. If no assumption other than its triangularity is made about the triangle $ABC$, then the symbol “$ABC$” can be taken as denoting any triangle one pleases. Then the geometer’s argument establishes that any triangle has the attribute in question, and if any triangle has it, then all triangles do. We now introduce a notation analogous to the geometer’s in talking about “any arbitrarily selected triangle $ABC$.” This will avoid the pretense of listing an indefinite or infinite number of substitution instances of a propositional function, for instead we shall talk about any substitution instance of the propositional function.

We shall use the (hitherto unused) lowercase letter $y$ to denote any arbitrarily selected individual. We shall use it in a way similar to that in which the geometer used the letters $ABC$. Because the truth of any substitution instance of a propositional function follows from its universal quantification, we can infer the substitution instance that results from replacing $x$ by $y$, where $y$ denotes “any arbitrarily selected” individual. Thus we may begin our formal proof of the validity of the given argument as follows:

1. $\forall x (Hx \supset Mx)$
2. $\forall x (Gx \supset Hx)$
   $\therefore \forall x (Gx \supset Mx)$
3. $Hy \supset My$ 1, U.I.
4. $Gy \supset Hy$ 2, U.I.
5. $Gy \supset My$ 4,3, H.S.

From the premises we have deduced the statement $Gy \supset My$, which in effect, because $y$ denotes “any arbitrarily selected individual,” asserts the truth of any substitution instance of the propositional function $Gx \supset Mx$. Because any substitution instance is true, all substitution instances must be true, and hence the universal quantification of that propositional function is true also. We may add this principle to our list of rules of inference, stating it as follows: From the substitution instance of a propositional function with respect to the name of any arbitrarily selected individual, one can validly infer the universal quantification of that propositional function. This new principle permits us to generalize, that is, to go from a special substitution instance to a generalized or universally quantified expression, so we refer to it as the principle of Universal Generalization and abbreviate it as U.G. It is stated as

$$U.G.: \quad \Phi y \quad \therefore \quad \forall x (\Phi x)$$

(where $y$ denotes “any arbitrarily selected individual”)
The sixth and final line of the formal proof already begun may now be written (and justified) as

6. \((x)(Gx \supset Mx)\) 5, U.G.

Let us review the preceding discussion. In the geometer’s proof, the only assumption made about \(ABC\) is that it is a triangle; hence what is proved true of \(ABC\) is proved true of any triangle. In our proof, the only assumption made about \(y\) is that it is an individual; hence what is proved true of \(y\) is proved true of any individual. The symbol \(y\) is an individual symbol, but it is a very special one. Typically it is introduced into a proof by using U.I., and only the presence of \(y\) permits the use of U.G.

Here is another valid argument, the demonstration of whose validity requires the use of U.G. as well as U.I.: “No humans are perfect. All Greeks are humans. Therefore no Greeks are perfect.* The formal proof of its validity is:

1. \((x)(Hx \supset \sim Px)\)
2. \((x)(Gx \supset Hx)\)
   \[\therefore (x)(Gx \supset \sim Px)\]
3. \(Hy \supset \sim Py\) 1, U.I.
4. \(Gy \supset Hy\) 2, U.I.
5. \(Gy \supset \sim Py\) 4,3, H.S.
6. \((x)(Gx \supset \sim Px)\) 5, U.G.

There may seem to be some artificiality about the preceding. It may be urged that distinguishing carefully between \((x)(\Phi x)\) and \(\Phi y\), so that they are not treated as identical but must be inferred from each other by U.I. and U.G., is to insist on a distinction without a difference. But there certainly is a formal difference between them. The statement \((x)(Hx \supset Mx)\) is a noncompound statement, whereas \(Hy \supset My\) is compound, being a conditional. From the two noncompound statements \((x)(Gx \supset Hx)\) and \((x)(Hx \supset Mx)\), no relevant inference can be drawn by means of the original list of nineteen rules of inference. But from the compound statements \(Gy \supset Hy\) and \(Hy \supset My\), the indicated conclusion \(Gy \supset My\) follows by a hypothetical syllogism. The principle of U.I. is used to get from noncompound statements, to which our earlier rules of inference do not

---

*This is an appropriate point to observe that, for arguments of some kinds, the traditional syllogistic analysis can establish validity as efficiently as modern quantified logic. A classical logician would quickly identify this syllogism as having the mood EAE in the first figure—necessarily of the form Celarent, and therefore immediately seen to be valid. See Section 6.5 for a summary exposition of the valid standard-form categorical syllogisms.
usefully apply, to compound statements, to which they can be applied to derive the desired conclusion. The quantification principles thus augment our logical apparatus to make it capable of validating arguments essentially involving noncompound (generalized) propositions as well as the other (simpler) kind of argument discussed in earlier chapters. On the other hand, in spite of this formal difference, there must be a logical equivalence between \( \forall x (\Phi x) \) and \( \Phi y \), or the rules U.I. and U.G. would not be valid. Both the difference and the logical equivalence are important for our purpose of validating arguments by reference to a list of rules of inference. The addition of U.I. and U.G. to our list strengthens it considerably.

The list must be expanded further when we turn to arguments that involve existential propositions. A convenient example with which to begin is “All criminals are vicious. Some humans are criminals. Therefore some humans are vicious.” It is symbolized as

\[
(\forall x)(Cx \supset Vx) \\
(\exists x)(Hx \land Cx) \\
\therefore (\exists x)(Hx \land Vx)
\]

The existential quantification of a propositional function is true if and only if it has at least one true substitution instance. Hence whatever attribute may be designated by \( \Phi \), \( (\exists x)(\Phi x) \) says that there is at least one individual that has the attribute \( \Phi \). If an individual constant (other than the special symbol \( y \)) is used nowhere earlier in the context, we may use it to denote either the individual that has the attribute \( \Phi \), or some one of the individuals that have \( \Phi \) if there are several. Knowing that there is such an individual, say, \( a \), we know that \( \Phi a \) is a true substitution instance of the propositional function \( \Phi x \). Hence we add to our list of rules of inference this principle: \textit{From the existential quantification of a propositional function, we may infer the truth of its substitution instance with respect to any individual constant (other than \( y \)) that occurs nowhere earlier in that context.} The new rule of inference is the principle of \textbf{Existential Instantiation} and is abbreviated as “E.I.” It is stated as:

\[
\text{E.I.:} \\
(\exists x)(\Phi x) \\
\therefore \Phi v \\
\text{[where } v \text{ is any individual constant (other than } y \text{) having no previous occurrence in the context]}
\]

 Granted the additional rule of inference E.I., we may begin a demonstration of the validity of the stated argument:

\[
1. \ (\forall x)(Cx \supset Vx) \\
2. \ (\exists x)(Hx \land Cx) \\
\therefore (\exists x)(Hx \land Vx)
\]
Thus far we have deduced $Ha \cdot Va$, which is a substitution instance of the propositional function whose existential quantification is asserted by the conclusion. Because the existential quantification of a propositional function is true if and only if it has at least one true substitution instance, we add to our list of rules of inference the principle that from any true substitution instance of a propositional function we may validly infer the existential quantification of that propositional function. This fourth and final rule of inference is the principle of **Existential Generalization**, abbreviated as E.G. and stated as

$$\exists v \cdot \Phi v \vdash (\exists x)(\Phi x) \quad (\text{where } v \text{ is any individual symbol})$$

The tenth and final line of the demonstration already begun may now be written (and justified) as

$$10. (\exists x)(Hx \cdot Vx) \quad 9, \text{ E.G.}$$

The need for the indicated restriction on the use of E.I. can be seen by considering the obviously invalid argument, “Some alligators are kept in captivity. Some birds are kept in captivity. Therefore some alligators are birds.” If we failed to heed the restriction on E.I. that a substitution instance of a propositional function inferred by E.I. from the existential quantification of that propositional function can contain only an individual symbol (other than $y$) that has no previous occurrence in the context, then we might proceed to construct a “proof” of validity for this invalid argument. Such an erroneous “proof” might proceed as follows:

1. $(\exists x)(Ax \cdot Cx)$
2. $(\exists x)(Bx \cdot Cx) 
   \vdash (\exists x)(Ax \cdot Bx)$
3. $Aa \cdot Ca \quad 1, \text{ E.I.}$
4. $Ba \cdot Ca \quad 2, \text{ E.I.} \quad (\text{wrong!})$
5. $Aa \quad 3, \text{ Simp.}$
6. $Ba \quad 4, \text{ Simp.}$
7. $Aa \cdot Ba \quad 5,6, \text{ Conj.}$
8. $(\exists x)(Ax \cdot Bx) \quad 7, \text{ E.G.}$
The error in this “proof” occurs at line 4. From the second premise \((\exists x)(Bx \land Cx)\), we know that there is at least one thing that is both a bird and kept in captivity. If we were free to assign it the name \(a\) in line 4, we could, of course, assert \(Ba \land Ca\). But we are not free to make any such assignment of \(a\), for it has already been preempted in line 3 to serve as name for an alligator that is kept in captivity. To avoid errors of this sort, we must obey the indicated restriction whenever we use E.I. The preceding discussion should make clear that in any demonstration requiring the use of both E.I. and U.I., E.I. should always be used first.

For more complicated modes of argumentation, especially those that involve relations, certain additional restrictions must be placed on our four quantification rules. But for arguments of the present sort, traditionally called categorical syllogisms, the present restrictions are sufficient to prevent mistakes.

### OVERVIEW

#### Rules of Inference: Quantification

<table>
<thead>
<tr>
<th>Name</th>
<th>Abbreviation</th>
<th>Form</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Universal Instantiation</td>
<td>U.I.</td>
<td>((x)(\Phi x))</td>
<td>Any substitution instance of a propositional function can be validly inferred from its universal quantification.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(. (\nu))</td>
<td>(where (\nu) is any individual symbol)</td>
</tr>
<tr>
<td>Universal Generalization</td>
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</tr>
<tr>
<td></td>
<td></td>
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<td>(where (y) denotes “any arbitrarily selected individual”)</td>
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<td>E.I.</td>
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<td>(. (\exists x)(\Phi x))</td>
<td>(where (\nu) is any individual constant)</td>
</tr>
</tbody>
</table>
A. Construct a formal proof of validity for each of the following arguments.

**EXAMPLE**

1. \((x)(Ax \supset \sim Bx)\)  
   \((\exists x)(Cx \bullet Ax)\)  
   \[\therefore (\exists x)(Cx \bullet \sim Bx)\]

**SOLUTION**

The conclusion of this argument is an existentially quantified statement. Plainly, the last step will therefore be the application of E.G. To obtain the line needed, we will first have to instantiate the premises, applying E.I. to the second premise and U.I. to the first premise. The restriction on the use of E.I. makes it essential that we apply E.I. before we apply U.I., so that we may use the same individual constant, say \(a\), for both. The proof looks like this:

1. \((x)(Ax \supset \sim Bx)\)
2. \((\exists x)(Cx \bullet Ax)\)  
   \[\therefore (\exists x)(Cx \bullet \sim Bx)\]
3. \(Ca \bullet Aa\)  
   2, E.I.
4. \(Aa \supset \sim Ba\)  
   1, U.I.
5. \(Aa \bullet Ca\)  
   3, Com.
6. \(Aa\)  
   5, Simp.
7. \(\sim Ba\)  
   4, 6, M.P.
8. \(Ca\)  
   3, Simp.
9. \(Ca \bullet \sim Ba\)  
   8, 7, Conj.
10. \((\exists x)(Cx \bullet \sim Bx)\)  
    9, E.G.
B. Construct a formal proof of validity for each of the following arguments, in each case using the suggested notations.

1. No athletes are bookworms. Carol is a bookworm. Therefore Carol is not an athlete. 
   \((Ax \supset \neg Bx)\) 

2. All dancers are exuberant. Some fencers are not exuberant. Therefore some fencers are not dancers. 
   \((Dx \supset Ex) \quad (Fx \supset \neg Ex) \quad \therefore (Fx \supset \neg Dx)\) 

3. No gamblers are happy. Some idealists are happy. Therefore some idealists are not gamblers. 
   \((Gx \supset Hx) \quad (Ix \supset \neg Hx) \quad \therefore (Ix \supset \neg Gx)\) 

4. All jesters are knaves. No knaves are lucky. Therefore no jesters are lucky. 
   \((Jx \cdot Kx)\) 

5. All mountaineers are neighborly. Some outlaws are mountaineers. Therefore some outlaws are neighborly. 
   \((Mx \supset Nx)\) 

6. Only pacifists are Quakers. There are religious Quakers. Therefore pacifists are sometimes religious. 
   \((Px \cdot Qx)\) 

7. To be a swindler is to be a thief. None but the underprivileged are thieves. Therefore swindlers are always underprivileged. 
   \((Sx \supset Tx) \quad (\neg Tx \supset \neg Sx) \quad \therefore Sx \supset Tx\) 

8. No violinists are not wealthy. There are no wealthy xylophonists. Therefore violinists are never xylophonists. 
   \((Vx \supset Wx) \quad (Wx \supset \neg Xx) \quad \therefore (Vx \supset \neg Xx)\) 

9. None but the brave deserve the fair. Only soldiers are brave. Therefore the fair are deserved only by soldiers. 
   \((Dx: x \text{ deserves the fair}; \quad Bx: x \text{ is brave}; \quad Sx: x \text{ is a soldier})\) 

10. All mountaineers are neighborly. Some outlaws are mountaineers. Therefore some outlaws are neighborly. 
    \((Mx \supset Nx)\) 

11. All jesters are knaves. No knaves are lucky. Therefore no jesters are lucky. 
    \((Jx \cdot Kx)\) 

*1. No athletes are bookworms. Carol is a bookworm. Therefore Carol is not an athlete. \((Ax, Bx, c)\) 

*2. All dancers are exuberant. Some fencers are not exuberant. Therefore some fencers are not dancers. \((Dx, Ex, Fx)\) 

*3. No gamblers are happy. Some idealists are happy. Therefore some idealists are not gamblers. \((Gx, Hx, Ix)\) 

*4. All jesters are knaves. No knaves are lucky. Therefore no jesters are lucky. \((Jx, Kx, Lx)\)
10. Everyone that asketh receiveth. Simon receiveth not. Therefore Simon asketh not. (Ax, Rx, s)

10.6 Proving Invalidity

To prove the invalidity of an argument involving quantifiers, we can use the method of refutation by logical analogy. For example, the argument, “All conservatives are opponents of the administration; some delegates are opponents of the administration; therefore some delegates are conservatives,” is proved invalid by the analogy, “All cats are animals; some dogs are animals; therefore some dogs are cats;” which is obviously invalid, because its premises are known to be true and its conclusion known to be false. Such analogies, however, are not always easy to devise. Some more effective method of proving invalidity is desirable.

In Chapter 9 we developed a method of proving invalidity for arguments involving truth-functional compound statements. That method consisted of making truth-value assignments to the component simple statements in arguments, in such a way as to make the premises true and the conclusions false. That method can be adapted for arguments involving quantifiers. The adaptation involves our general assumption that there is at least one individual. For an argument involving quantifiers to be valid, it must be impossible for its premises to be true and its conclusion false as long as there is at least one individual.

The general assumption that there is at least one individual is satisfied if there is exactly one individual, or exactly two individuals, or exactly three individuals, or . . . . If any one of these assumptions about the exact number of individuals is made, there is an equivalence between general propositions and truth-functional compounds of singular propositions. If there is exactly one individual, say $a$, then

$$(x)(\Phi x) \equiv \Phi a \equiv (\exists x)(\Phi x)$$

If there are exactly two individuals, say $a$ and $b$, then

$$(x)(\Phi x) \equiv [\Phi a \cdot \Phi b] \quad \text{and} \quad (\exists x)(\Phi x) \equiv [\Phi a \lor \Phi b]$$

If there are exactly three individuals, say $a$, $b$, and $c$, then

$$(x)(\Phi x) \equiv [\Phi a \cdot \Phi b \cdot \Phi c] \quad \text{and} \quad (\exists x)(\Phi x) \equiv [\Phi a \lor \Phi b \lor \Phi c]$$

In general, if there are exactly $n$ individuals, say $a$, $b$, $c$, . . . . $n$, then

$$(x)(\Phi x) \equiv [\Phi a \cdot \Phi b \cdot \Phi c \cdot \ldots \cdot \Phi n] \quad \text{and} \quad (\exists x)(\Phi x) \equiv [\Phi a \lor \Phi b \lor \Phi c \lor \ldots \lor \Phi n]$$
These biconditionals are true as a consequence of our definitions of the universal and existential quantifiers. No use is made here of the four quantification rules explained in Section 10.5.

An argument involving quantifiers is valid if, and only if, it is valid no matter how many individuals there are, provided there is at least one. So an argument involving quantifiers is proved invalid if there is a possible universe or model containing at least one individual such that the argument’s premises are true and its conclusion false of that model. Consider the argument, “All mercenaries are undependable. No guerrillas are mercenaries. Therefore no guerrillas are undependable.” It may be symbolized as

\[(\forall x)(Mx \supset Ux)\]
\[(\forall x)(Gx \supset \sim Mx)\]
\[\therefore (\forall x)(Gx \supset \sim Ux)\]

If there is exactly one individual, say \(a\), this argument is logically equivalent to

\[Ma \supset Ua\]
\[Ga \supset \sim Ma\]
\[\therefore Ga \supset \sim Ua\]

The latter can be proved invalid by assigning the truth value true to \(Ga\) and \(Ua\) and false to \(Ma\). (This assignment of truth values is a shorthand way of describing the model in question as containing only the one individual, \(a\), which is a guerrilla and undependable but is not a mercenary.) Hence the original argument is not valid for a model containing exactly one individual, and it is therefore invalid. Similarly, we can prove the invalidity of the first argument mentioned in this section (on p. 462) by describing a model containing exactly one individual, \(a\), so that \(Aa\) and \(Da\) are assigned truth and \(Ca\) is assigned falsehood.*

Some arguments, for example,

\[(\exists x)Fx\]
\[\therefore (x)Fx\]

*Here we assume that the simple predicates \(Ax, Bx, Cx, Dx, \ldots\), occurring in our propositions are neither necessary, that is, logically true of all individuals (for example, \(x\) is identical with itself), nor impossible, that is, logically false of all individuals (for example, \(x\) is different from itself). We also assume that the only logical relations among the simple predicates involved are those asserted or logically implied by the premises. The point of these restrictions is to permit us to assign truth values arbitrarily to the substitution instances of these simple predicates without any inconsistency—for of course, a correct description of any model must be consistent.
may be valid for any model in which there is exactly one individual, but invalid for a model containing two or more individuals. Such arguments must also count as invalid, because a valid argument must be valid regardless of how many individuals there are, so long as there is at least one. Another example of this kind of argument is “All collies are affectionate. Some collies are watchdogs. Therefore all watchdogs are affectionate.” Its symbolic translation is

\[
(x)(Cx \supset Ax) \\
(\exists x)(Cx \cdot Wx) \\
\therefore (x)(Wx \supset Ax)
\]

For a model containing exactly one individual, \(a\), it is logically equivalent to

\[
Ca \supset Aa \\
Ca \cdot Wa \\
\therefore Wa \supset Aa
\]

which is valid. But for a model containing two individuals, \(a\) and \(b\), it is logically equivalent to

\[
(Ca \supset Aa) \cdot (Cb \supset Ab) \\
(Ca \cdot Wa) \lor (Cb \cdot Wb) \\
\therefore (Wa \supset Aa) \cdot (Wb \supset Ab)
\]

which is proved invalid by assigning \textit{truth} to \(Ca, Aa, Wa, Wb\), and \textit{falsehood} to \(Cb\) and \(Ab\). Hence the original argument is not valid for a model containing exactly two individuals, and it is therefore \textit{invalid}. For any invalid argument of this general type, it is possible to describe a model containing some definite number of individuals for which its logically equivalent truth-functional argument can be proved invalid by the method of assigning truth values.

It should be emphasized again: In moving from a given argument involving general propositions to a truth-functional argument (one that is logically equivalent to the given argument for a specified model), no use is made of our four quantification rules. Instead, each statement of the truth-functional argument is logically equivalent to the corresponding general proposition of the given argument, and that logical equivalence is shown by the biconditionals formulated earlier in this section on page 463, whose logical truth for the model in question follows from the very definitions of the universal and existential quantifiers.

The procedure for proving the invalidity of an argument containing general propositions is the following. First, consider a one-element model containing only the individual \(a\). Then, write out the logically equivalent truth-functional argument for that model, which is obtained by moving from each
A general proposition (quantified propositional function) of the original argument to the substitution instance of that propositional function with respect to \( a \). If the truth-functional argument can be proved invalid by assigning truth values to its component simple statements, that suffices to prove the original argument invalid. If that cannot be done, next consider a two-element model containing the individuals \( a \) and \( b \). In order to obtain the logically equivalent truth-functional argument for this larger model, one can simply join to each original substitution instance with respect to \( a \) to a new substitution instance of the same propositional function with respect to \( b \). This “joining” must be in accord with the logical equivalences stated on page 463; that is, where the original argument contains a universally quantified propositional function, \((\forall x)(\Phi x)\), the new substitution instance \( \Phi b \) is combined with the first substitution instance \( a \) by conjunction (“\( \& \)”); but where the original argument contains an existentially quantified propositional function \((\exists x)(\Phi x)\), the new substitution instance \( \Phi b \) is combined with the first substitution instance \( a \) by disjunction (“\( \lor \)”). The preceding example illustrates this procedure. If the new truth-functional argument can be proved invalid by assigning truth values to its component simple statements, that suffices to prove the original argument invalid. If that cannot be done, next consider a three-element model containing the individuals \( a, b, \) and \( c \). And so on. None of the exercises in this book requires a model containing more than three elements.

**EXERCISES**

In the following exercises, no model containing more than two elements is required.

A. Prove the invalidity of the following:

**EXAMPLE**

1. \((\exists x)(Ax \land Bx)\)
   \((\exists x)(Cx \land Bx)\)
   \(\therefore (\exists x)(Cx \lor \neg Ax)\)

**SOLUTION**

We first construct a model (or possible universe) containing exactly one individual, \( \mathbb{U} \). We then exhibit the logically equivalent propositions in that model. Thus,

\[
\begin{align*}
(\exists x)(Ax \land Bx) \\
(\exists x)(Cx \land Bx) \quad \text{logically equivalent to} \quad \{ Aa \land Ba \} \\
\therefore (x)(Cx \lor \neg Ax) \quad \text{in } \mathbb{U} \\
\quad \text{in } \mathbb{U} \quad \{ Ca \land Ba \} \\
\therefore Ca \lor \neg Aa
\end{align*}
\]
We may prove the argument invalid in this model by assigning truth values as follows:

Because the argument has been proved invalid in this model, the argument has been proved invalid.

1. \( \exists x (Ax \land Bx) \)
   \( \exists x (Cx \land Bx) \)
   \( \therefore (\exists x (Cx \supset \neg Ax)) \)

2. \( (\exists x (Dx \supset \neg Ex)) \)
   \( (\exists x (Ex \supset Fx)) \)
   \( \therefore (\exists x (Fx \supset \neg Dx)) \)

3. \( (\exists x (Gx \supset Hx)) \)
   \( (\exists x (Gx \supset Ix)) \)
   \( \therefore (\exists x (Ix \supset Hx)) \)

4. \( (\exists x (Jx \land Kx)) \)
   \( (\exists x (Kx \land Lx)) \)
   \( \therefore (\exists x (Lx \land Jx)) \)

5. \( (\exists x (Mx \land Nx)) \)
   \( (\exists x (Mx \land Ox)) \)
   \( \therefore (\exists x (Ox \supset Nx)) \)

6. \( (\exists x (Px \supset \neg Qx)) \)
   \( (\exists x (Px \supset \neg Rx)) \)
   \( \therefore (\exists x (Rx \supset \neg Qx)) \)

7. \( (\exists x (Sx \supset \neg Tx)) \)
   \( (\exists x (Tx \supset Ux)) \)
   \( \therefore (\exists x (Ux \supset \neg Sx)) \)

8. \( (\exists x (Vx \land \neg Wx)) \)
   \( (\exists x (Wx \supset \neg Xx)) \)
   \( \therefore (\exists x (Xx \land \neg Vx)) \)

9. \( (\exists x (Yx \land Zx)) \)
   \( (\exists x (Ax \land Zx)) \)
   \( \therefore (\exists x (Ax \supset \neg Yx)) \)

10. \( (\exists x (Bx \land \neg Cx)) \)
    \( (\exists x (Ax \supset \neg Cx)) \)
    \( \therefore (\exists x (Ax \supset \neg Bx)) \)

B. Prove the invalidity of the following, in each case using the suggested notation.

*1. All anarchists are bearded. All communists are bearded. Therefore all anarchists are communists. \( (Ax, Bx, Cx) \)

2. No diplomats are extremists. Some fanatics are extremists. Therefore some diplomats are not fanatics. \( (Dx, Ex, Fx) \)

3. All generals are handsome. Some intellectuals are handsome. Therefore some generals are intellectuals. \( (Gx, Hx, Ix) \)

4. Some journalists are not kibitzers. Some kibitzers are not lucky. Therefore some journalists are not lucky. \( (Jx, Kx, Lx) \)

*5. Some malcontents are noisy. Some officials are not noisy. Therefore no officials are malcontents. \( (Mx, Nx, Ox) \)

6. Some physicians are quacks. Some quacks are not responsible. Therefore some physicians are not responsible. \( (Px, Qx, Rx) \)

7. Some politicians are leaders. Some leaders are not orators. Therefore some orators are not politicians. \( (Px, Lx, Ox) \)
8. None but the brave deserve the fair. Every soldier is brave. Therefore none but soldiers deserve the fair. \((Dx: x \text{ deserves the fair}; Bx: x \text{ is brave}; Sx: x \text{ is a soldier})\)

9. If anything is metallic, then it is breakable. There are breakable ornaments. Therefore there are metallic ornaments. \((Mx, Bx, Ox)\)

*10. Only students are members. Only members are welcome. Therefore all students are welcome. \((Sx, Mx, Wx)\)

### 10.7 Asylogistic Inference

All the arguments considered in the preceding two sections were of the form traditionally called *categorical syllogisms*. These consist of two premises and a conclusion, each of which is analyzable either as a singular proposition or as one of the A, E, I, or O varieties. We turn now to the problem of evaluating somewhat more complicated arguments. These require no greater logical apparatus than has already been developed, yet they are *asylogistic arguments*; that is, they cannot be reduced to standard-form categorical syllogisms, and therefore evaluating them requires a more powerful logic than was traditionally used in testing categorical syllogisms.

In this section we are still concerned with general propositions, formed by quantifying propositional functions that contain only a single individual variable. In the categorical syllogism, the only kinds of propositional functions quantified were of the forms \(\neg \Psi x\), \(\Phi x \lor \Psi x\), \(\Phi x \land \neg \Phi x\), and \(\Phi x \land \Sigma \Phi x\). Now we shall be quantifying propositional functions with more complicated internal structures. An example will help make this clear. Consider the argument

Hotels are both expensive and depressing.
Some hotels are shabby.
Therefore some expensive things are shabby.

This argument, for all its obvious validity, is not amenable to the traditional sort of analysis. True enough, it could be expressed in terms of A and I propositions by using the symbols \(Hx, Bx, Sx, \text{ and } Ex\) to abbreviate the propositional functions “\(x\) is a hotel,” “\(x\) is both expensive and depressing,” “\(x\) is shabby,” and “\(x\) is expensive,” respectively.* Using these abbreviations, we might propose to symbolize the given argument as

\[\begin{align*}
(x)(Hx \land Bx) \\
(\exists x)(Hx \land Sx) \\
\therefore (\exists x)(Ex \land Sx)
\end{align*}\]

*This would, however, violate the restriction stated in the footnote on page 464.
Forcing the argument into the straitjacket of the traditional A and I forms in this way obscures its validity. The argument just given in symbols is invalid, although the original argument is perfectly valid. A notation restricted to categorical propositions here obscures the logical connection between Bx and Ex. A more adequate analysis is obtained by using Hx, Sx, and Ex, as explained, plus Dx as an abbreviation for “x is depressing.” By using these symbols, the original argument can be translated as

1. $(\forall x)(Hx \supset (Ex \cdot Dx))$
2. $(\exists x)(Hx \cdot Sx)$
   \[\therefore (\exists x)(Ex \cdot Sx)\]

Thus symbolized, a demonstration of its validity is easily constructed. One such demonstration proceeds as follows:

3. $Hw \cdot Sw$  
   2, E.I.
4. $Hw \supset (Ew \cdot Dw)$  
   1, U.I.
5. $Hw$  
   3, Simp.
6. $Ew \cdot Dw$  
   4,5, M.P.
7. $Ew$  
   6, Simp.
8. $Sw \cdot Hw$  
   3, Com.
9. $Sw$  
   8, Simp.
10. $Ew \cdot Sw$  
    7,9, Conj.
11. $(\exists x)(Ex \cdot Sx)$  
    10, E.G.

In symbolizing general propositions that result from quantifying more complicated propositional functions, care must be taken not to be misled by the deceptiveness of ordinary English. One cannot translate from English into our logical notation by following any formal or mechanical rules. In every case, one must understand the meaning of the English sentence, and then symbolize that meaning in terms of propositional functions and quantifiers.

Three locutions of ordinary English that are sometimes troublesome are the following. First, note that a statement such as “All athletes are either very strong or very quick” is not a disjunction, although it contains the connective “or.” It definitely does not have the same meaning as “Either all athletes are very strong or all athletes are very quick.” The former is properly symbolized—using obvious abbreviations—as

$$(x)[Ax \supset (Sx \lor Qx)]$$

whereas the latter is symbolized as

$$(x)(Ax \supset Sx) \lor (x)(Ax \supset Qx)$$
Second, note that a statement such as “Oysters and clams are delicious”—while it can be stated as the conjunction of two general propositions, “Oysters are delicious and clams are delicious”—also can be stated as a single noncompound general proposition; in which case the word “and” is properly symbolized by the “∨” rather than by the “•.” The stated proposition is symbolized as

\( (x)[(Ox ∨ Cx) ⊃ Dx] \)

not as

\( (x)[(Ox • Cx) ⊃ Dx] \)

For to say that oysters and clams are delicious is to say that anything is delicious that is either an oyster or a clam, not to say that anything is delicious that is both an oyster and a clam.

Third, what are called exceptive propositions require very careful attention. Such propositions—for example, “All except previous winners are eligible”—may be treated as the conjunction of two general propositions. Using the example just given, we might reasonably understand the proposition to assert both that previous winners are not eligible, and that those who are not previous winners are eligible. It is symbolized as:

\( (x)(Px ⊃ ~Ex) • (x)(~Px ⊃ Ex) \)

The same exceptive proposition may also be translated as a noncompound general proposition that is the universal quantification of a propositional function containing the symbol for material equivalence “≡,” a biconditional, and symbolized thus

\( (x)(Ex ≡ ~Px) \)

which can also be rendered in English as “Anyone is eligible if and only if that person is not a previous winner.” In general, exceptive propositions are most conveniently regarded as quantified biconditionals.

Whether a proposition is in fact exceptive is sometimes difficult to determine. A recent controversy requiring resolution by a federal court panel illustrates this contextual difficulty. The Census Act, a law that establishes the rules for the conduct of the national census every ten years, contains the following passage:

Sec. 195. Except for the determination of population for purposes of apportionment of Representatives in Congress among the several States, the Secretary [of Commerce] shall, if he considers it feasible, authorize the use of the statistical method known as “sampling” in carrying out the provisions of this title.
For the 2000 census, which did determine population for the purposes of apportionment, the Census Bureau sought to use the sampling technique, and was sued by the House of Representatives, which claimed that the passage quoted here prohibits sampling in such a census. The Bureau defended its plan, contending that the passage authorizes the use of sampling in some contexts, but in apportionment contexts leaves the matter undetermined. Which interpretation of that exceptive provision in the statute is correct?

The court found the House position correct, writing:

Consider the directive “except for my grandmother’s wedding dress, you shall take the contents of my closet to the cleaners.” It is . . . likely that the granddaughter would be upset if the recipient of her directive were to take the wedding dress to the cleaners and subsequently argue that she had left this decision to his discretion. The reason for this result . . . is because of our background knowledge concerning wedding dresses: We know they are extraordinarily fragile and of deep sentimental value to family members. We therefore would not expect that a decision to take [that] dress to the cleaners would be purely discretionary.

The apportionment of Congressional representatives among the states is the wedding dress in the closet. . . . The apportionment function is the “sole constitutional function of the decennial enumeration.” The manner in which it is conducted may impact not only the distribution of representatives among the states, but also the balance of political power within the House. . . . This court finds that the Census Act prohibits the use of statistical sampling to determine the population for the purpose of apportionment of representatives among the states. . . .* The exceptive proposition in this statute is thus to be understood as asserting the conjunction of two propositions: (1) that the use of sampling is not permitted in the context of apportionment, and (2) that in all other contexts sampling is discretionary. A controversial sentence in exceptive form must be interpreted in its context.

In Section 10.5, our list of rules of inference was expanded by four, and we showed that the expanded list was sufficient to demonstrate the validity of categorical syllogisms when they are valid. And we have just seen that the same expanded list suffices to establish the validity of asyllogistic arguments of the type described. Now we may observe that, just as the expanded list was sufficient to establish validity in asyllogistic arguments, so also the method of proving syllogisms invalid (explained in Section 10.6) by describing possible nonempty

*Decided by a specially appointed Voting Rights Act panel of three judges on 24 August 1998.
universes, or models, is sufficient to prove the invalidity of asyllogistic arguments of the present type as well. The following asyllogistic argument,

Managers and superintendents are either competent workers or relatives of the owner.

Anyone who dares to complain must be either a superintendent or a relative of the owner.

Managers and foremen alone are competent workers.

Someone did dare to complain.

Therefore some superintendent is a relative of the owner.

may be symbolized as

\[(x)[(Mx \lor Sx) \supset (Cx \lor Rx)]\]

\[(x)[Dx \supset (Sx \lor Rx)]\]

\[(x)(Mx = Cx)\]

\[(\exists x)Dx\]

\[\therefore (\exists x)(Sx \land Rx)\]

and we can prove it invalid by describing a possible universe or model containing the single individual \(a\) and assigning the truth value true to \(Ca, Da, Fa, Ra\), and the truth value false to \(Sa\).

**EXERCISES**

A. Translate the following statements into logical symbolism, in each case using the abbreviations suggested.

**EXAMPLE**

1. Apples and oranges are delicious and nutritious. \((Ax, Ox, Dx, Nx)\)

**SOLUTION**

The meaning of this proposition clearly is that if anything is either an apple or an orange it is both delicious and nutritious. Hence it is symbolized as

\[(x)[(Ax \lor Ox) \supset (Dx \land Nx)]\]

2. Some foods are edible only if they are cooked. \((Fx, Ex, Cx)\)

3. No car is safe unless it has good brakes. \((Cx, Sx, Bx)\)
4. Any tall man is attractive if he is dark and handsome. \((Tx, Mx, Ax, Dx, Hx)\)

*5. A gladiator wins if and only if he is lucky. \((Gx, Wx, Lx)\)

6. A boxer who wins if and only if he is lucky is not skillful. \((Bx, Wx, Lx, Sx)\)

7. Not all people who are wealthy are both educated and cultured. \((Px, Wx, Ex, Cx)\)

8. Not all tools that are cheap are either soft or breakable. \((Tx, Cx, Sx, Bx)\)

9. Any person is a coward who deserts. \((Px, Cx, Dx)\)

*10. To achieve success, one must work hard if one goes into business, or study continuously if one enters a profession. \((Ax: x\text{ achieves success}; Wx: x\text{ works hard}; Bx: x\text{ goes into business}; Sx: x\text{ studies continuously}; Px: x\text{ enters a profession})\)

11. An old European joke goes like this: In America, everything is permitted that is not forbidden. In Germany, everything is forbidden that is not permitted. In France, everything is permitted even if it’s forbidden. In Russia, everything is forbidden even if it’s permitted. \((Ax: x\text{ is in America}; Gx: x\text{ is in Germany}; Fx: x\text{ is in France}; Rx: x\text{ is in Russia}; Px: x\text{ is permitted}; Nx: x\text{ is forbidden})\)

B. For each of the following, either construct a formal proof of validity or prove it invalid. If it is to be proved invalid, a model containing as many as three elements may be required.

*1. \((\forall x)[(Ax \lor Bx) \supset (Cx \land Dx)] \therefore (\forall x)(Bx \supset Cx)\)

2. \((\exists x)[(Ex \land Fx) \land [(Ex \lor Fx) \supset (Gx \land Hx)]] \therefore (\exists x)(Ex \supset Hx)\)

3. \((\exists x)[(Jx \land ~Kx)] \land [(Jx \supset (Ix \land Kx))] \land (\exists x)[(Ix \land Jx) \land ~Lx] \therefore (\exists x)(Kx \land Lx)\)

4. \((\exists x)[(Mx \land Nx) \supset (Ox \lor Px)]\)

\((\exists x)[(Ox \land Px) \supset (Qx \lor Rx)]\)

\((\exists x)[(Mx \lor Ox) \supset Rx]\)

*5. \((\exists x)(Sx \land Tx)\)

\((\exists x)(Ux \land ~Sx)\)

\((\exists x)(Vx \land ~Tx)\)

\((\exists x)(Ux \land Vx)\)
6. \( (x)[Wx \supset (Xx \supset Yx)] \)
   \( (\exists x)[Xx \cdot (Zx \cdot \sim Ax)] \)
   \( (x)[Wx \supset Yx] \supset (Bx \supset Ax)] \)
   \( \therefore (\exists x)[Zx \cdot \sim Bx] \)

7. \( (\exists x)[Cx \cdot \sim (Dx \supset Ex)] \)
   \( (x)[(Cx \cdot Dx) \supset Fx] \)
   \( (\exists x)[Ex \cdot \sim (Dx \supset Cx)] \)
   \( (x)[Gx \supset Cx] \)
   \( \therefore (\exists x)[Gx \cdot \sim Fx] \)

8. \( (x)[Hx \supset lx] \)
   \( (x)[(Hx \cdot lx) \supset Jx] \)
   \( (x)[\sim Kx \supset (Hx \lor lx)] \)
   \( (x)[(Jx \lor \sim Jx) \supset (lx \supset Hx)] \)
   \( \therefore (x)[Jx \lor Kx] \)

9. \( (x)[(Lx \lor Mx) \supset [(Nx \cdot Ox) \lor Px] \supset Qx}] \)
   \( (\exists x)[Mx \cdot \sim Lx] \)
   \( (x)[(Ox \lor Qx) \cdot \sim Rx] \supset Mx] \)
   \( (\exists x)[Lx \cdot \sim Mx] \)
   \( \therefore (\exists x)[Nx \supset Rx] \)

10. \( (x)[(Sx \lor Tx) \supset \sim (Ux \lor Vx)] \)
    \( (\exists x)[Sx \cdot \sim Wx] \)
    \( (\exists x)[Tx \cdot \sim Xx] \)
    \( (x)[\sim Wx \supset Xx] \)
    \( \therefore (\exists x)[Ux \cdot \sim Vx] \)

C. For each of the following, either construct a formal proof of its validity or prove it invalid, in each case using the suggested notation.

*1. Acids and bases are chemicals. Vinegar is an acid. Therefore vinegar is a chemical. \((Ax, Bx, Cx, Vx)\)

2. Teachers are either enthusiastic or unsuccessful. Teachers are not all unsuccessful. Therefore there are enthusiastic teachers. \((Tx, Ex, Ux)\)

3. Argon compounds and sodium compounds are either oily or volatile. Not all sodium compounds are oily. Therefore some argon compounds are volatile. \((Ax, Sx, Ox, Vx)\)

4. No employee who is either slovenly or discourteous can be promoted. Therefore no discourteous employee can be promoted. \((Ex, Sx, Dx, Px)\)
5. No employer who is either inconsiderate or tyrannical can be successful. Some employers are inconsiderate. There are tyrannical employers. Therefore no employer can be successful. (Ex, Ix, Tx, Sx)

6. There is nothing made of gold that is not expensive. No weapons are made of silver. Not all weapons are expensive. Therefore not everything is made of gold or silver. (Gx, Ex, Wx, Sx)

7. There is nothing made of tin that is not cheap. No rings are made of lead. Not everything is either tin or lead. Therefore not all rings are cheap. (Tx, Cx, Rx, Lx)

8. Some prize fighters are aggressive but not intelligent. All prize fighters wear gloves. Prize fighters are not all aggressive. Any slugger is aggressive. Therefore not every slugger wears gloves. (Px, Ax, Ix, Gx, Sx)

9. Some photographers are skillful but not imaginative. Only artists are photographers. Photographers are not all skillful. Any journeyman is skillful. Therefore not every artist is a journeyman. (Px, Sx, Ix, Ax, Jx)

10. A book is interesting only if it is well written. A book is well written only if it is interesting. Therefore any book is both interesting and well written if it is either interesting or well written. (Bx, Ix, Wx)

D. Do the same (as in Set C) for each of the following.

1. All citizens who are not traitors are present. All officials are citizens. Some officials are not present. Therefore there are traitors. (Cx, Tx, Px, Ox)

2. Doctors and lawyers are professional people. Professional people and executives are respected. Therefore doctors are respected. (Dx, Lx, Px, Ex, Rx)

3. Only lawyers and politicians are members. Some members are not college graduates. Therefore some lawyers are not college graduates. (Lx, Px, Mx, Cx)

4. All cut-rate items are either shopworn or out of date. Nothing shopworn is worth buying. Some cut-rate items are worth buying. Therefore some cut-rate items are out of date. (Cx, Sx, OX, Wx)

5. Some diamonds are used for adornment. Only things worn as jewels or applied as cosmetics are used for adornment. Diamonds are never applied as cosmetics. Nothing worn as a jewel is properly used if it has an industrial application. Some diamonds have industrial applications. Therefore some diamonds are not properly used. (Dx, Ax, Jx, Cx, Px, Ix)
6. No candidate who is either endorsed by labor or opposed by the *Tribune* can carry the farm vote. No one can be elected who does not carry the farm vote. Therefore no candidate endorsed by labor can be elected. 

\( (C_x, L_x, O_x, F_x, E_x) \)

7. No metal is friable that has been properly tempered. No brass is properly tempered unless it is given an oil immersion. Some of the ash trays on the shelf are brass. Everything on the shelf is friable. Brass is a metal. Therefore some of the ash trays were not given an oil immersion. 

\( (M_x: x \text{ is metal}; \ F_x: x \text{ is friable}; \ T_x: x \text{ is properly tempered}; \ B_x: x \text{ is brass}; \ O_x: x \text{ is given an oil immersion}; \ A_x: x \text{ is an ash tray}; \ S_x: x \text{ is on the shelf}) \)

8. Anyone on the committee who knew the nominee would vote for the nominee if free to do so. Everyone on the committee was free to vote for the nominee except those who were either instructed not to by the party caucus or had pledged support to someone else. Everyone on the committee knew the nominee. No one who knew the nominee had pledged support to anyone else. Not everyone on the committee voted for the nominee. Therefore the party caucus had instructed some members of the committee not to vote for the nominee. 

\( (C_x: x \text{ is on the committee}; \ K_x: x \text{ knows the nominee}; \ V_x: x \text{ votes for the nominee}; \ F_x: x \text{ is free to vote for the nominee}; \ I_x: x \text{ is instructed by the party caucus not to vote for the nominee}; \ P_x: x \text{ had pledged support to someone else}) \)

9. All logicians are deep thinkers and effective writers. To write effectively, one must be economical if one’s audience is general, and comprehensive if one’s audience is technical. No deep thinker has a technical audience if he has the ability to reach a general audience. Some logicians are comprehensive rather than economical. Therefore not all logicians have the ability to reach a general audience. 

\( (L_x: x \text{ is a logician}; \ D_x: x \text{ is a deep thinker}; \ W_x: x \text{ is an effective writer}; \ E_x: x \text{ is economical}; \ G_x: x \text{’s audience is general}; \ C_x: x \text{ is comprehensive}; \ T_x: x \text{’s audience is technical}; \ A_x: x \text{ has the ability to reach a general audience}) \)

\*10. Some criminal robbed the Russell mansion. Whoever robbed the Russell mansion either had an accomplice among the servants or had to break in. To break in, one would either have to smash the door or pick the lock. Only an expert locksmith could have picked the lock. Had anyone smashed the door, he would have been heard. Nobody was heard. If the criminal who robbed the Russell
mansion managed to fool the guard, he must have been a convincing actor. No one could rob the Russell mansion unless he fooled the guard. No criminal could be both an expert locksmith and a convincing actor. Therefore some criminal had an accomplice among the servants. (Cx: x is a criminal; Rx: x robbed the Russell mansion; Sx: x had an accomplice among the servants; Bx: x broke in; Dx: x smashed the door; Px: x picked the lock; Lx: x is an expert locksmith; Hx: x was heard; Fx: x fooled the guard; Ax: x is a convincing actor)

11. If anything is expensive it is both valuable and rare. Whatever is valuable is both desirable and expensive. Therefore if anything is either valuable or expensive then it must be both valuable and expensive. (Ex: x is expensive; Vx: x is valuable; Rx: x is rare; Dx: x is desirable)

12. Figs and grapes are healthful. Nothing healthful is either illaudable or jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore some figs are illaudable. (Fx: x is a fig; Gx: x is a grape; Hx: x is healthful; Ix: x is illaudable; Jx: x is jejune; Kx: x is knurly)

13. Figs and grapes are healthful. Nothing healthful is both illaudable and jejune. Some grapes are jejune and knurly. Some figs are not knurly. Therefore some figs are not illaudable. (Fx: x is a fig; Gx: x is a grape; Hx: x is healthful; Ix: x is illaudable; Jx: x is jejune; Kx: x is knurly)

14. Gold is valuable. Rings are ornaments. Therefore gold rings are valuable ornaments. (Gx: x is gold; Vx: x is valuable; Rx: x is a ring; Ox: x is an ornament)

*15. Oranges are sweet. Lemons are tart. Therefore oranges and lemons are sweet or tart. (Ox: x is an orange; Sx: x is sweet; Lx: x is a lemon; Tx: x is tart)

16. Socrates is mortal. Therefore everything is either mortal or not mortal. (s: Socrates; Mx: x is mortal)

SUMMARY

In Section 10.1, we explained that the analytical techniques of the previous chapters are not adequate to deal with arguments whose validity depends on the inner logical structure of noncompound propositions. We described quantification in general terms as a theory that, with some additional
symbolization, enables us to exhibit this inner structure and thereby greatly
enhances our analytical powers.

In Section 10.2, we explained singular propositions and introduced the
symbols for an individual variable \(x\), for individual constants (lowercase
letters \(a\) through \(w\)), and for attributes (capital letters). We introduced the
concept of a propositional function, an expression that contains an individual
variable and becomes a statement when an individual constant is substituted
for the individual variable. A proposition may thus be obtained from a proposi-
tional function by the process of instantiation.

In Section 10.3, we explained how propositions also can be obtained from
propositional functions by means of generalization, that is, by the use of quan-
tifiers such as “everything,” “nothing,” and “some.” We introduced the uni-
versal quantifier \(\forall x\), meaning “given any \(x\),” and the existential quantifier
\(\exists x\), meaning “there is at least one \(x\) such that.” On a square of opposition, we
showed the relations between universal and existential quantification.

In Section 10.4, we showed how each of the four main types of general
propositions,

- **A**: universal affirmative propositions
- **E**: universal negative propositions
- **I**: particular affirmative propositions
- **O**: particular negative propositions

is correctly symbolized by propositional functions and quantifiers. We also explained the modern interpretation of the relations of **A**, **E**, **I**, and **O**
propositions.

In Section 10.5, we expanded the list of rules of inference, adding four
additional rules:

- Universal Instantiation, U.I.
- Universal Generalization, U.G.
- Existential Instantiation, E.I.
- Existential Generalization, E.G.

and showed how, by using these and the other nineteen rules set forth earlier,
we can construct a formal proof of validity of deductive arguments that
depend on the inner structure of noncompound propositions.

In Section 10.6, we explained how the method of refutation by logical
analogy can be used to prove the invalidity of arguments involving quanti-
fiers by creating a model, or possible universe, containing exactly one, or
exactly two, or exactly three (etc.) individuals and the restatement of the con-
stituent propositions of an argument in that possible universe. An argument
involving quantifiers is proved invalid if we can exhibit a possible universe containing at least one individual, such that the argument’s premises are true and its conclusion is false in that universe.

In Section 10.7, we explained how we can symbolize and evaluate asyllo-gistic arguments, those containing propositions not reducible to A, E, I, and O propositions, or singular propositions. We noted the complexity of exceptive propositions and other propositions whose logical meaning must first be understood and then rendered accurately with propositional functions and quantifiers.