9.5 Constructing More Extended Formal Proofs

Arguments whose formal proof requires only two additional statements are quite simple. We now advance to construct formal proofs of the validity of more complex arguments. However, the process will be the same: The target for the final statement of the sequence will always be the conclusion of the argument, and the rules of inference will always be our only logical tools.

Let us look closely at an example—the first exercise of Set A on page 387, an argument whose proof requires three additional statements:

\[
1. A \lor (B \supset A) \\
\text{~}A \land C \\
\therefore \text{~}B
\]

In devising the proof of this argument (as in most cases), we need some plan of action, some strategy with which we can progress, using our rules,
toward the conclusion sought. Here that conclusion is \( \sim B \). We ask ourselves: Where in the premises does \( B \) appear? Only as the antecedent of the hypothetical (\( B \supset A \)), which is a component of the first premise. How might \( \sim B \) be derived? Using Modus Tollens, we can infer it from \( B \supset A \) if we can establish that hypothetical separately and also establish \( \sim A \). Both of those needed steps can be readily accomplished. \( \sim A \) is inferred from line 2 by Simplification:

3. \( \sim A \) 2, Simp.

We can then apply \( \sim A \) to line 1, using Disjunctive Syllogism to infer (\( B \supset A \)):

4. \( (B \supset A) \) 1, 3, D.S.

The proof may then be completed using Modus Tollens on lines 4 and 3:

5. \( \sim B \) 4, 3, M.T.

The strategy used in this argument is readily devised. In the case of some proofs, devising the needed strategy will not be so simple, but it is almost always helpful to ask: What statement(s) will enable one to infer the conclusion? And what statement(s) will enable one to infer that? And so on, moving backward from the conclusion toward the premises given.

**EXERCISES**

A. For each of the following arguments, it is possible to provide a formal proof of validity by adding just three statements to the premises. Writing these out, carefully and accurately, will strengthen your command of the rules of inference, a needed preparation for the construction of proofs that are more extended and more complex.

1. \( A \lor (B \supset A) \)
   \( \sim A \land C \)
   \( \therefore \sim B \)

2. \( (D \lor E) \supset (F \land G) \)
   \( D \)
   \( \therefore F \)

3. \( (H \supset I) \land (H \supset J) \)
   \( H \land (I \lor J) \)
   \( \therefore I \lor J \)

4. \( (K \land L) \supset M \)
   \( K \supset L \)
   \( \therefore K \supset [(K \land L) \land M] \)

5. \( N \supset [(N \supset O) \supset P] \)
   \( N \land O \)
   \( \therefore P \)

*5. \( N \supset [(N \supset O) \supset P] \)

6. \( Q \supset R \)
   \( R \supset S \)
   \( \sim S \)
   \( \therefore \sim Q \land \sim R \)

7. \( T \supset U \)
   \( V \lor \sim U \)
   \( \sim V \land \sim W \)
   \( \therefore \sim T \)

8. \( \sim X \supset Y \)
   \( Z \supset X \)
   \( \sim X \)
   \( \therefore Y \land \sim Z \)
Formal proofs most often require more than two or three lines to be added to the premises. Some are very lengthy. Whatever their length, however, the same process and the same strategic techniques are called for in devising the needed proofs. In this section we rely entirely on the nine elementary valid argument forms that serve as our rules of inference.

As we begin to construct longer and more complicated proofs, let us look closely at an example of such proofs—the first exercise of Set B on page 389. It is not difficult, but it is more extended than those we have worked with so far.

1. \( A \lor B \)
2. \( A \lor (C \cdot D) \)
3. \( \lnot B \cdot \lnot E \)
4. \( \therefore C \)

The strategy needed for the proof of this argument is not hard to see: To obtain \( C \) we must break apart the premise in line 2; to do that we will need \( \lnot A \); to establish \( \lnot A \) we will need to apply Modus Tollens to line 1 using \( \lnot B \). Therefore we continue the sequence with the fourth line of the proof by applying Simplification to line 3:

1. \( A \lor B \)
2. \( A \lor (C \cdot D) \)
3. \( \lnot B \cdot \lnot E \quad /\cdot \quad C \)
4. \( \lnot B \quad 3, \text{Simp.} \)
Using line 4 we can obtain \( \neg A \) from line 1:

5. \( \neg A \) 1, 4, M.T.

With \( \neg A \) established we can break line 2 apart, as we had planned, using D.S.:

6. \( C \cdot D \) 2, 5, D.S.

The conclusion may be pulled readily from the sixth line by Simplification.

7. \( C \) 6, Simp.

Seven lines (including the premises) are required for this formal proof. Some proofs require very many more lines than this, but the object and the method remain always the same.

It sometimes happens, as one is devising a formal proof, that a statement is correctly inferred and added to the numbered sequence but turns out not to be needed; a solid proof may be given without using that statement. In such a case it is usually best to rewrite the proof, eliminating the unneeded statement. However, if the unneeded statement is retained, and the proof remains accurately constructed using other statements correctly inferred, the inclusion of the unneeded statement (although perhaps inelegant) does not render the proof incorrect. Logicians tend to prefer shorter proofs, proofs that move to the conclusion as directly as the rules of inference permit. But if, as one is constructing a more complicated proof, it becomes apparent that some much earlier statement(s) has been needlessly inferred, it may be more efficient to allow such statement(s) to remain in place, using (as one goes forward) the more extended numbering that that inclusion makes necessary. Logical solidity is the critical objective. A solid formal proof, one in which each step is correctly derived and the conclusion is correctly linked to the premises by an unbroken chain of arguments using the rules of inference correctly, remains a proof—even if it is not as crisp and elegant as some other proof that could be devised.

**EXERCISES**

B. For each of the following arguments, a formal proof of validity can be constructed without great difficulty, although some of the proofs may require a sequence of eight or nine lines (including premises) for their completion.

1. \( A \supset B \)
   \( A \lor (C \cdot D) \)
   \( \neg B \cdot \neg E \)
   \( \therefore C \)

2. \( (F \supset G) \cdot (H \supset I) \)
   \( J \supset K \)
   \( (F \lor J) \cdot (H \lor L) \)
   \( \therefore G \lor K \)

3. \( (\neg M \cdot \neg N) \supset (O \supset N) \)
   \( N \supset M \)
   \( \neg M \)
   \( \therefore \neg O \)

4. \( (K \lor L) \supset (M \lor N) \)
   \( (M \lor N) \supset (O \cdot P) \)
   \( K \)
   \( \therefore O \)
In the study of logic, our aim is to evaluate arguments in a natural language, such as English. When an argument in everyday discourse confronts us, we can prove it to be valid (if it really is valid) by first translating the statements (from English, or from any other natural language) into our symbolic language, and then constructing a formal proof of that symbolic translation. The symbolic version of the argument may reveal that the argument is, in fact, more simple (or possibly more complex) than one had supposed on first hearing or reading it. Consider the following example (the first in the set of exercises that immediately follow):

1. If either Gertrude or Herbert wins, then both Jens and Kenneth lose. Gertrude wins. Therefore Jens loses. (G—Gertrude wins; H—Herbert wins; J—Jens loses; K—Kenneth loses.)

Abbreviations for each statement are provided in this context because, without them, those involved in the discussion of these arguments would be likely to employ various abbreviations, making communication difficult. Using the abbreviations suggested greatly facilitates discussion.

Translated from the English into symbolic notation, this first argument appears as

1. \((G \lor H) \supset (J \land K)\)
2. \(G\) \hspace{1cm} \therefore J

The formal proof of this argument is short and straightforward:

3. \(G \lor H\) \hspace{1cm} 2, Add.
4. \(J \land K\) \hspace{1cm} 1, 3, M. P.
5. \(J\) \hspace{1cm} 4, Simp.
EXERCISES

C. Each of the following arguments in English may be similarly translated, and for each, a formal proof of validity (using only the nine elementary valid argument forms as rules of inference) may be constructed. These proofs vary in length, some requiring a sequence of thirteen statements (including the premises) to complete the formal proofs. The suggested abbreviations should be used for the sake of clarity. Bear in mind that, as one proceeds to produce a formal proof of an argument presented in a natural language, it is of the utmost importance that the translation into symbolic notation of the statements appearing discursively in the argument be perfectly accurate; if it is not, one will be working with an argument that is different from the original one, and in that case any proof devised will be useless, being not applicable to the original argument.

1. If either Gertrude or Herbert wins, then both Jens and Kenneth lose.
   Gertrude wins. Therefore Jens loses. (G—Gertrude wins; H—Herbert wins; J—Jens loses; K—Kenneth loses.)

2. If Adriana joins, then the club’s social prestige will rise; and if Boris
   joins, then the club’s financial position will be more secure. Either
   Adriana or Boris will join. If the club’s social prestige rises, then Boris
   will join; and if the club’s financial position becomes more secure, then
   Wilson will join. Therefore either Boris or Wilson will join. (A—Adriana
   joins; S—The club’s social prestige rises; B—Boris joins; F—The club’s
   financial position is more secure; W—Wilson joins.)

3. If Brown received the message, then she took the plane; and if she took
   the plane, then she will not be late for the meeting. If the message was
   incorrectly addressed, then Brown will be late for the meeting. Either
   Brown received the message or the message was incorrectly addressed.
   Therefore either Brown took the plane or she will be late for the meeting.
   (R—Brown received the message; P—Brown took the plane; L—Brown
   will be late for the meeting; T—The message was incorrectly addressed.)

4. If Nihar buys the lot, then an office building will be constructed; whereas
   if Payton buys the lot, then it will be quickly sold again. If Rivers buys
   the lot, then a store will be constructed; and if a store is constructed, then
   Thompson will offer to lease it. Either Nihar or Rivers will buy the lot.
   Therefore either an office building or a store will be constructed. (N—
   Nihar buys the lot; O—An office building will be constructed; P—Payton
   buys the lot; Q—The lot will be quickly sold again; R—Rivers buys the
   lot; S—A store will be constructed; T—Thompson will offer to lease it.)

*5. If rain continues, then the river rises. If rain continues and the river
   rises, then the bridge will wash out. If the continuation of rain would
cause the bridge to wash out, then a single road is not sufficient for the town. Either a single road is sufficient for the town or the traffic engineers have made a mistake. Therefore the traffic engineers have made a mistake. (C—Rain continues; R—The river rises; B—The bridge washes out; S—A single road is sufficient for the town; M—The traffic engineers have made a mistake.)

6. If Jonas goes to the meeting, then a complete report will be made; but if Jonas does not go to the meeting, then a special election will be required. If a complete report is made, then an investigation will be launched. If Jonas’s going to the meeting implies that a complete report will be made, and the making of a complete report implies that an investigation will be launched, then either Jonas goes to the meeting and an investigation is launched or Jonas does not go to the meeting and no investigation is launched. If Jonas goes to the meeting and an investigation is launched, then some members will have to stand trial. But if Jonas does not go to the meeting and no investigation is launched, then the organization will disintegrate very rapidly. Therefore either some members will have to stand trial or the organization will disintegrate very rapidly. (J—Jonas goes to the meeting; R—A complete report is made; E—A special election is required; I—An investigation is launched; T—Some members have to stand trial; D—The organization disintegrates very rapidly.)

7. If Ann is present, then Bill is present. If Ann and Bill are both present, then either Charles or Doris will be elected. If either Charles or Doris is elected, then Elmer does not really dominate the club. If Ann’s presence implies that Elmer does not really dominate the club, then Florence will be the new president. So Florence will be the new president. (A—Ann is present; B—Bill is present; C—Charles will be elected; D—Doris will be elected; E—Elmer really dominates the club; F—Florence will be the new president.)

8. If Mr. Jones is the manager’s next-door neighbor, then Mr. Jones’s annual earnings are exactly divisible by 3. If Mr. Jones’s annual earnings are exactly divisible by 3, then $40,000 is exactly divisible by 3. But $40,000 is not exactly divisible by 3. If Mr. Robinson is the manager’s next-door neighbor, then Mr. Robinson lives halfway between Detroit and Chicago. If Mr. Robinson lives in Detroit, then he does not live halfway between Detroit and Chicago. Mr. Robinson lives in Detroit. If Mr. Jones is not the manager’s next-door neighbor, then either Mr. Robinson or Mr. Smith is the manager’s next-door neighbor. Therefore Mr. Smith is the manager’s next-door neighbor. (J—Mr. Jones
is the manager’s next-door neighbor; \(E\)—Mr. Jones’s annual earnings are exactly divisible by 3; \(T\)—$40,000 is exactly divisible by 3; \(R\)—Mr. Robinson is the manager’s next-door neighbor; \(H\)—Mr. Robinson lives halfway between Detroit and Chicago; \(D\)—Mr. Robinson lives in Detroit; \(S\)—Mr. Smith is the manager’s next-door neighbor.)

9. If Mr. Smith is the manager’s next-door neighbor, then Mr. Smith lives halfway between Detroit and Chicago. If Mr. Smith lives halfway between Detroit and Chicago, then he does not live in Chicago. Mr. Smith is the manager’s next-door neighbor. If Mr. Robinson lives in Detroit, then he does not live in Chicago. Mr. Robinson lives in Detroit. Mr. Smith lives in Chicago or else either Mr. Robinson or Mr. Jones lives in Chicago. If Mr. Jones lives in Chicago, then the manager is Jones. Therefore the manager is Jones. (\(S\)—Mr. Smith is the manager’s next-door neighbor; \(W\)—Mr. Smith lives halfway between Detroit and Chicago; \(L\)—Mr. Smith lives in Chicago; \(D\)—Mr. Robinson lives in Detroit; \(I\)—Mr. Robinson lives in Chicago; \(C\)—Mr. Jones lives in Chicago; \(B\)—The manager is Jones.)

*10. If Smith once beat the editor at billiards, then Smith is not the editor. Smith once beat the editor at billiards. If the manager is Jones, then Jones is not the editor. The manager is Jones. If Smith is not the editor and Jones is not the editor, then Robinson is the editor. If the manager is Jones and Robinson is the editor, then Smith is the publisher. Therefore Smith is the publisher. (\(O\)—Smith once beat the editor at billiards; \(M\)—Smith is the editor; \(B\)—The manager is Jones; \(N\)—Jones is the editor; \(F\)—Robinson is the editor; \(G\)—Smith is the publisher.)

9.6 Expanding the Rules of Inference: Replacement Rules

The nine elementary valid argument forms with which we have been working are powerful tools of inference, but they are not powerful enough. There are very many valid truth-functional arguments whose validity cannot be proved using only the nine rules thus far developed. We need to expand the set of rules, to increase the power of our logical toolbox.

To illustrate the problem, consider the following simple argument, which is plainly valid:

If you travel directly from Chicago to Los Angeles, you must cross the Mississippi River. If you travel only along the Atlantic seaboard, you will not cross the Mississippi River. Therefore if you travel directly from Chicago to Los Angeles, you will not travel only along the Atlantic seaboard.