9

Methods of Deduction

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9.1 Formal Proof of Validity

In theory, truth tables are adequate to test the validity of any argument of the general type we have considered. In practice, however, they become unwieldy as the number of component statements increases. A more efficient method of establishing the validity of an extended argument is to deduce its conclusion from its premises by a sequence of elementary arguments, each of which is known to be valid. This technique accords fairly well with ordinary methods of argumentation.

Consider, for example, the following argument:

If Anderson was nominated, then she went to Boston.

If she went to Boston, then she campaigned there.

If she campaigned there, she met Douglas.

Anderson did not meet Douglas.

Either Anderson was nominated or someone more eligible was selected.

Therefore someone more eligible was selected.

The validity of this argument may be intuitively obvious, but let us consider the matter of proof. The discussion will be facilitated by translating the argument into symbolism as
To establish the validity of this argument by means of a truth table requires a table with thirty-two rows, because five different simple statements are involved. We can prove the argument valid by deducing its conclusion instead using a sequence of just four elementary valid arguments. From the first two premises, $A \supset B$ and $B \supset C$, we validly infer that $A \supset C$ as a Hypothetical Syllogism. From $A \supset C$ and the third premise, $C \supset D$, we validly infer that $A \supset D$ as another Hypothetical Syllogism. From $A \supset D$ and the fourth premise, $\sim D$, we validly infer that $\sim A$ by Modus Tollens. And from $\sim A$ and the fifth premise, $A \lor E$, as a Disjunctive Syllogism we validly infer $E$, the conclusion of the original argument. That the conclusion can be deduced from the five premises of the original argument by four elementary valid arguments proves the original argument to be valid. Here the elementary valid argument forms Hypothetical Syllogism (H.S.), Modus Tollens (M.T.), and Disjunctive Syllogism (D.S.) are used as rules of inference in accordance with which conclusions are validly inferred or deduced from premises.

This method of deriving the conclusion of a deductive argument—using rules of inference successively to prove the validity of the argument—is as reliable as the truth-table method discussed in Chapter 8, if the rules are used with meticulous care. But it improves on the truth-table method in two ways: It is vastly more efficient, as has just been shown; and it enables us to follow the flow of the reasoning process from the premises to the conclusion and is therefore much more intuitive and more illuminating. The method is often called natural deduction. Using natural deduction, we can provide a formal proof of the validity of an argument that is valid.

A formal proof of validity is given by writing the premises and the statements that we deduce from them in a single column, and setting off in another column, to the right of each such statement, its “justification,” or the reason we give for including it in the proof. It is convenient to list all the premises first and to write the conclusion either on a separate line, or slightly to one side and separated by a diagonal line from the premises. If all the statements in the column are numbered, the “justification” for each statement consists of the numbers of the preceding statements from which it is inferred, together
with the abbreviation for the rule of inference by which it follows from them.

The formal proof of the example argument is written as

1. $A \supset B$
2. $B \supset C$
3. $C \supset D$
4. $\sim D$
5. $A \lor E$
   \[\therefore E\]
6. $A \supset C$ \hspace{1cm} 1, 2, H.S.
7. $A \supset D$ \hspace{1cm} 6, 3, H.S.
8. $\sim A$ \hspace{1cm} 7, 4, M.T.
9. $E$ \hspace{1cm} 5, 8, D.S.

We define a **formal proof of validity** of a given argument as a sequence of statements, each of which is either a premise of that argument or follows from preceding statements of the sequence by an elementary valid argument, such that the last statement in the sequence is the conclusion of the argument whose validity is being proved.

We define an **elementary valid argument** as any argument that is a substitution instance of an elementary valid argument form. Note that any substitution instance of an elementary valid argument form is an elementary valid argument. Thus the argument

$$(A \land B) \supset [C = (D \lor E)]$$

$A \land B$
\[\therefore C = (D \lor E)\]

is an elementary valid argument because it is a substitution instance of the elementary valid argument form *Modus Ponens* (M.P.). It results from

$$p \supset q$$

$$p$$
\[\therefore q\]

by substituting $A \land B$ for $p$ and $C = (D \lor E)$ for $q$, and it is therefore of that form even though *modus ponens* is not the specific form of the given argument.

*Modus Ponens* is a very elementary valid argument form indeed, but what other valid argument forms are considered to be rules of inference? We begin with a list of just nine rules of inference that can be used in constructing formal proofs of validity. With their aid, formal proofs of validity can be constructed for a wide range of more complicated arguments. The names provided are for the most part standard, and the use of their abbreviations permits formal proofs to be set down with a minimum of writing.