24. Egypt's food shortage worsens, but Saudi Arabia buys five hundred more warplanes and Libya raises the price of oil.

***25.** Libya raises the price of oil and Egypt's food shortage worsens; however, Saudi Arabia buys five hundred more warplanes and Jordan requests more U.S. aid.

8.3 Conditional Statements and Material Implication

Where two statements are combined by placing the word "if" before the first and inserting the word "then" between them, the resulting compound statement is a **conditional** statement (also called a *hypothetical*, an *implication*, or an *implicative statement*). In a conditional statement the component statement that follows the "if" is called the **antecedent** (or the *implicans* or—rarely—the *protasis*), and the component statement that follows the "then" is the **consequent** (or the *implicate* or—rarely—the *apodosis*). For example, "If Mr. Jones is the brakeman's next-door neighbor, then Mr. Jones earns exactly three times as much as the brakeman" is a conditional statement in which "Mr. Jones is the brakeman's next-door neighbor" is the antecedent and "Mr. Jones earns exactly three times as much as the brakeman" is the consequent.

A conditional statement asserts that in any case in which its antecedent is true, its consequent is also true. It does not assert that its antecedent is true, but only that *if* its antecedent is true, then its consequent is also true. It does not assert that its consequent is true, but only that its consequent is true *if* its antecedent is true. The essential meaning of a conditional statement is the *relationship* asserted to hold between the antecedent and the consequent, in that order. To understand the meaning of a conditional statement, then, we must understand what the relationship of implication is.

Implication plausibly appears to have more than one meaning. We found it useful to distinguish different senses of the word "or" before introducing a special logical symbol to correspond exactly to a single one of the meanings of the English word. Had we not done so, the ambiguity of the English would have infected our logical symbolism and prevented it from achieving the clarity and precision aimed at. It will be equally useful to distinguish the different senses of "implies" or "if–then" before we introduce a special logical symbol in this connection.

Consider the following four conditional statements, each of which seems to assert a different type of implication, and to each of which corresponds a different sense of "if–then":

A. If all humans are mortal and Socrates is a human, then Socrates is mortal.

B. If Leslie is a bachelor, then Leslie is unmarried.

C. If this piece of blue litmus paper is placed in acid, then this piece of blue litmus paper will turn red.

D. If State loses the homecoming game, then I'll eat my hat.

Even a casual inspection of these four conditional statements reveals that they are of quite different types. The consequent of **A** follows *logically* from its antecedent, whereas the consequent of **B** follows from its antecedent by the very *definition* of the term "bachelor," which means unmarried man. The consequent of **C** does not follow from its antecedent either by logic alone or by the definition of its terms; the connection must be discovered empirically, because the implication stated here is *causal*. Finally, the consequent of **D** does not follow from its antecedent either by logic or by definition, nor is there any causal law involved. Statement **D** reports a *decision* of the speaker to behave in the specified way under the specified circumstances.

These four conditional statements are different in that each asserts a different type of implication between its antecedent and its consequent. But they are not completely different; all assert types of implication. Is there any identifiable common meaning, any partial meaning that is common to these admittedly different types of implication, although perhaps not the whole or complete meaning of any one of them?

The search for a common partial meaning takes on added significance when we recall our procedure in working out a symbolic representation for the English word "or." In that case, we proceeded as follows. First, we emphasized the difference between the two senses of the word, contrasting inclusive with exclusive disjunction. The inclusive disjunction of two statements was observed to mean that at least one of the statements is true, and the exclusive disjunction of two statements was observed to mean that at least one of the statements is true but not both are true. Second, we noted that these two types of disjunction had a common *partial* meaning. This partial common meaning, that at least one of the disjuncts is true, was seen to be the *whole* meaning of the weak, inclusive "or," and a *part* of the meaning of the strong, exclusive "or." We then introduced the special symbol "∨" to represent this common partial meaning (which is the entire meaning of "or" in its inclusive sense). Third, we noted that the symbol representing the common partial meaning is an adequate translation of either sense of the word "or" for the purpose of retaining the disjunctive syllogism as a valid form of argument. It was admitted that translating an exclusive "or" into the symbol "∨" ignores and loses part of the word's meaning. But the part of its meaning that is preserved by this translation is all that is needed for the disjunctive syllogism to remain a valid form of argument. Because the disjunctive syllogism is typical of arguments involving disjunction, with which we are concerned here, this partial translation of the word "or," which may abstract from its "full" or "complete" meaning in some cases, is wholly adequate for our present purposes.

Now we wish to proceed in the same way, this time in connection with the English phrase "if–then." The first part is already accomplished: We have already emphasized the differences among four senses of the "if–then" phrase, corresponding to four different types of implication. We are now ready for the second step, which is to discover a sense that is at least a part of the meaning of all four types of implication.

We approach this problem by asking: What circumstances suffice to establish the falsehood of a given conditional statement? Under what circumstances should we agree that the conditional statement

> If this piece of blue litmus paper is placed in that acid solution, then this piece of blue litmus paper will turn red.

is *false?* It is important to realize that this conditional does not assert that any blue litmus paper is actually placed in the solution, or that any litmus paper actually turns red. It asserts merely that *if* this piece of blue litmus paper is placed in the solution, *then* this piece of blue litmus paper will turn red. It is proved false if this piece of blue litmus paper is actually placed in the solution and does not turn red. The acid test, so to speak, of the falsehood of a conditional statement is available when its antecedent is true, because if its consequent is false while its antecedent is true, the conditional itself is thereby proved false.

Any conditional statement, "If p, then q," is known to be false if the conjunction $p \bullet {\sim}q$ is known to be true—that is, if its antecedent is true and its consequent is false. For a conditional to be true, then, the indicated conjunction must be false; that is, its negation ${\sim}(p \bullet {\sim}q)$ must be true. In other words, for any conditional, "If p then q," to be true, ${\sim}(p \bullet {\sim}q)$, the negation of the conjunction of its antecedent with the negation of its consequent, must also be true. We may then regard ${\sim}(p \bullet {\sim}q)$ as a part of the meaning of "If p then q."

Every conditional statement means to deny that its antecedent is true and its consequent false, but this need not be the whole of its meaning. A conditional such as **A** on page 331 also asserts a logical connection between its antecedent and consequent, as **B** asserts a definitional connection, **C** a causal connection, and **D** a decisional connection. No matter what type of implication is asserted by a conditional statement, *part* of its meaning is the negation of the conjunction of its antecedent with the negation of its consequent.

We now introduce a special symbol to represent this common partial meaning of the "if–then" phrase. We define the new symbol "$\supset$," called a

horseshoe, by taking $p \supset q$ as an abbreviation of $\sim(p \bullet \sim q)$. The exact significance of the $\supset$ symbol can be indicated by means of a truth table:

p	q	$\sim q$	$p \bullet \sim q$	$\sim(p \bullet \sim q)$	$p \supset q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Here the first two columns are the guide columns; they simply lay out all possible combinations of truth and falsehood for p and q. The third column is filled in by reference to the second, the fourth by reference to the first and third, the fifth by reference to the fourth, and the sixth is identical to the fifth by definition.

The symbol $\supset$ is not to be regarded as denoting *the* meaning of "if–then," or standing for *the* relation of implication. That would be impossible, for there is no single meaning of "if–then"; there are several meanings. There is no unique relation of implication to be thus represented; there are several different implication relations. Nor is the symbol $\supset$ to be regarded as somehow standing for *all* the meanings of "if–then." These are all different, and any attempt to abbreviate all of them by a single logical symbol would render that symbol ambiguous—as ambiguous as the English phrase "if–then" or the English word "implication." The symbol $\supset$ is completely unambiguous. What $p \supset q$ abbreviates is $\sim(p \bullet \sim q)$, whose meaning is included in the meanings of each of the various kinds of implications considered but does not constitute the entire meaning of any of them.

We can regard the symbol $\supset$ as representing another kind of implication, and it will be expedient to do so, because a convenient way to read $p \supset q$ is "If p, then q." But it is not the same kind of implication as any of those mentioned earlier. It is called **material implication** by logicians. In giving it a special name, we admit that it is a special notion, not to be confused with other, more usual, types of implication.

Not all conditional statements in English need assert one of the four types of implication previously considered. Material implication constitutes a fifth type that may be asserted in ordinary discourse. Consider the remark, "If Hitler was a military genius, then I'm a monkey's uncle." It is quite clear that it does not assert logical, definitional, or causal implication. It cannot represent a decisional implication, because it scarcely lies in the speaker's power to make the consequent true. No "real connection," whether logical, definitional, or causal, obtains between antecedent and consequent here. A conditional of this sort is often used as an emphatic or humorous method of denying its antecedent. The consequent of such a conditional is usually a statement that is obviously or ludicrously false. And because no true conditional can have both its antecedent true and its consequent false, to affirm such a conditional amounts to denying that its antecedent is true. The full meaning of the present conditional seems to be the denial that "Hitler was a military genius" is true when "I'm a monkey's uncle" is false. And because the latter is so obviously false, the conditional must be understood to deny the former.

The point here is that no "real connection" between antecedent and consequent is suggested by a material implication. All it asserts is that, as a matter of fact, it is not the case that the antecedent is true when the consequent is false. Note that the material implication symbol is a truth-functional connective, like the symbols for conjunction and disjunction. As such, it is defined by the truth table:

p	*q*	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

As thus defined by the truth table, the symbol ⊃ has some features that may at first appear odd: The assertion that a false antecedent materially implies a true consequent is true; and the assertion that a false antecedent materially implies a false consequent is also true. This apparent strangeness can be dissipated in part by the following considerations. Because the number 2 is smaller than the number 4 (a fact notated symbolically as $2 < 4$), it follows that *any* number smaller than 2 is smaller than 4. The conditional formula

$$\text{If } x < 2, \text{ then } x < 4.$$

is true for any number x whatsoever. If we focus on the numbers 1, 3, and 4, and replace the number variable x in the preceding conditional formula by each of them in turn, we can make the following observations. In

$$\text{If } 1 < 2, \text{ then } 1 < 4.$$

both antecedent and consequent are true, and of course the conditional is true. In

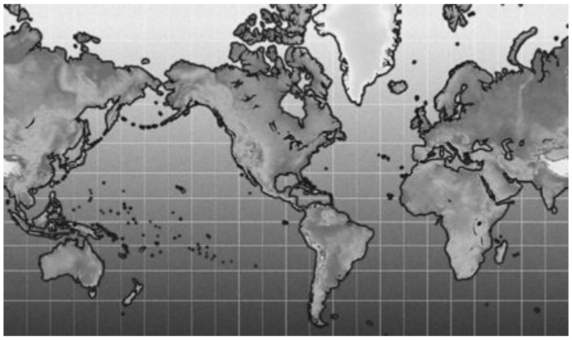
$$\text{If } 3 < 2, \text{ then } 3 < 4.$$

the antecedent is false and the consequent is true, and of course the conditional is again true. In

$$\text{If } 4 < 2, \text{ then } 4 < 4.$$

VISUAL LOGIC

Material Implication



Source: Photodisc/Getty Images

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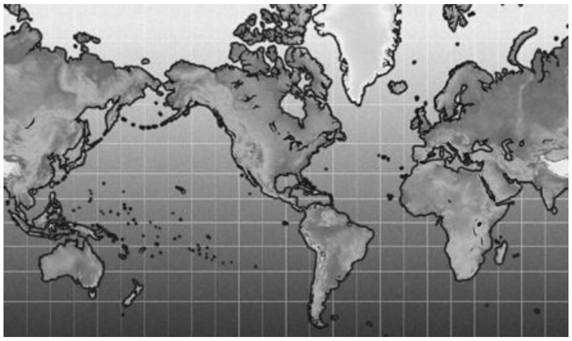
If the world is flat, then the moon is made of green cheese.

This proposition, in the form $F \supset G$, is a material implication. A material implication is true when the antecedent (the "if" clause) is false. Therefore a material implication is true when the antecedent is false and the consequent is also false, as in this illustrative proposition.



Source: Photodisc/Getty Images

Source: Photodisc/Getty Images

If the world is flat, the moon is round.

This proposition, in the similar form $F \supset R$, is also a material implication. A material implication is true when the antecedent (the "if" clause) is false. Therefore a material implication is true when the antecedent is false and the consequent is true, as in this illustrative proposition.

A material implication is false only if the antecedent is true and the consequent is false. Therefore a material implication is true whenever the antecedent is false, whether the consequent is false or true.

both antecedent and consequent are false, but the conditional remains true. These three cases correspond to the first, third, and fourth rows of the table defining the symbol ⊃. So it is not particularly remarkable or surprising that a conditional should be true when both antecedent and consequent are true, when the antecedent is false and the consequent is true, or when antecedent and consequent are both false. Of course, there is no number that is smaller than 2 but not smaller than 4; that is, there is no true conditional statement with a true antecedent and a false consequent. This is exactly what the defining truth table for ⊃ lays down.

Now we propose to translate any occurrence of the "if–then" phrase into our logical symbol ⊃. This proposal means that in translating conditional statements into our symbolism, we treat them all as merely material implications. Of course, most conditional statements assert more than that a merely material implication holds between their antecedents and consequents. So our proposal amounts to suggesting that we ignore, or put aside, or "abstract from," part of the meaning of a conditional statement when we translate it into our symbolic language. How can this proposal be justified?

The previous proposal to translate both inclusive and exclusive disjunctions by means of the symbol ∨ was justified on the grounds that the validity of the disjunctive syllogism was preserved even if the additional meaning that attaches to the exclusive "or" was ignored. Our present proposal to translate all conditional statements into the merely material implication symbolized by ⊃ may be justified in exactly the same way. Many arguments contain conditional statements of various different kinds, but the validity of all valid arguments of the general type with which we will be concerned is preserved even if the additional meanings of their conditional statements are ignored. This remains to be proved, of course, and will occupy our attention in the next section.

Conditional statements can be formulated in a variety of ways. The statement

> If he has a good lawyer, then he will be acquitted.

can equally well be stated without the use of the word "then" as

> If he has a good lawyer, he will be acquitted.

The order of the antecedent and consequent can be reversed, provided that the "if" still directly precedes the antecedent, as

> He will be acquitted if he has a good lawyer.

It should be clear that, in any of the examples just given, the word "if" can be replaced by such phrases as "in case," "provided that," "given that," or "on condition that," without any change in meaning. Minor adjustments in the phrasings of antecedent and consequent permit such alternative phrasings of the same conditional as

That he has a good lawyer implies that he will be acquitted.

or

His having a good lawyer entails his acquittal.

A shift from active to passive voice accompanies a reversal of order of antecedent and consequent, yielding the logically equivalent

His being acquitted is implied (or entailed) by his having a good lawyer.

Any of these is symbolized as $L \supset A$.

The notions of necessary and sufficient conditions provide other formulations of conditional statements. For any specified event, many circumstances are necessary for it to occur. Thus, for a normal car to run, it is necessary that there be fuel in its tank, its spark plugs properly adjusted, its oil pump working, and so on. So if the event occurs, every one of the conditions necessary for its occurrence must have been fulfilled. Hence to say

That there is fuel in its tank is a necessary condition for the car to run.

can equally well be stated as

The car runs only if there is fuel in its tank.

which is another way of saying that

If the car runs then there is fuel in its tank.

Any of these is symbolized as $R \supset F$. In general, "q is a *necessary condition* for p" is symbolized as $p \supset q$. And, likewise, "*p only if q*" is also symbolized a $p \supset q$.

For a specified situation there may be many alternative circumstances, any one of which is sufficient to produce that situation. For a purse to contain more than a dollar, for example, it is sufficient for it to contain five quarters, or eleven dimes, or twenty-one nickels, and so on. If any one of these circumstances obtains, the specified situation will be realized. Hence, to say "That the purse contains five quarters is a sufficient condition for it to contain more than a dollar" is to say "If the purse contains five quarters then it contains more than a dollar." In general, "p is a *sufficient condition* for q" is symbolized as $p \supset q$.

To illustrate, recruiters for the Wall Street investment firm Goldman, Sachs (where annual bonuses are commonly in the millions) grill potential employees repeatedly. Those who survive the grilling are invited to the firm's offices for a full day of interviews, culminating in a dinner with senior

Goldman executives. As reported recently, "Agile brains and near-perfect grades are necessary but not sufficient conditions for being hired. Just as important is fitting in."[3]

If p is a sufficient condition for q, we have $p \supset q$, and q must be a necessary condition for p. If p is a necessary condition for q, we have $q \supset p$, and q must be a sufficient condition for p. Hence, if p is necessary *and* sufficient for q, then q is sufficient *and* necessary for p.

Not every statement containing the word "if" is a conditional. None of the following statements is a conditional: "There is food in the refrigerator if you want some," "Your table is ready, if you please," "There is a message for you if you're interested," "The meeting will be held even if no permit is obtained." The presence or absence of particular words is never decisive. In every case, one must understand what a given sentence means, and then restate that meaning in a symbolic formula.

EXERCISES

A. If A, B, and C are true statements and X, Y, and Z are false statements, determine which of the following are true, using the truth tables for the horseshoe, the dot, the wedge, and the curl.

*1. $A \supset B$
2. $A \supset X$
3. $B \supset Y$
4. $Y \supset Z$
*5. $(A \supset B) \supset Z$
6. $(X \supset Y) \supset Z$
7. $(A \supset B) \supset C$
8. $(X \supset Y) \supset C$
9. $A \supset (B \supset Z)$
*10. $X \supset (Y \supset Z)$
11. $[(A \supset B) \supset C] \supset Z$
12. $[(A \supset X) \supset Y] \supset Z$
13. $[A \supset (X \supset Y)] \supset C$
14. $[A \supset (B \supset Y)] \supset X$
*15. $[(X \supset Z) \supset C] \supset Y$
16. $[(Y \supset B) \supset Y] \supset Y$
17. $[(A \supset Y) \supset B] \supset Z$
18. $[(A \bullet X) \supset C] \supset [(A \supset C) \supset X]$
19. $[(A \bullet X) \supset C] \supset [(A \supset X) \supset C]$
*20. $[(A \bullet X) \supset Y] \supset [(X \supset A) \supset (A \supset Y)]$
21. $[(A \bullet X) \vee (\sim A \bullet \sim X)] \supset [(A \supset X) \bullet (X \supset A)]$
22. $\{[A \supset (B \supset C)] \supset [(A \bullet B) \supset C]\} \supset [(Y \supset B) \supset (C \supset Z)]$
23. $\{[(X \supset Y) \supset Z] \supset [Z \supset (X \supset Y)]\} \supset [(X \supset Z) \supset Y]$
24. $[(A \bullet X) \supset Y] \supset [(A \supset X) \bullet (A \supset Y)]$
*25. $[A \supset (X \bullet Y)] \supset [(A \supset X) \vee (A \supset Y)]$

B. If A and B are known to be true, and X and Y are known to be false, but the truth values of P and Q are not known, of which of the following statements can you determine the truth values?

*1. $P \supset A$
2. $X \supset Q$
3. $(Q \supset A) \supset X$
4. $(P \bullet A) \supset B$
*5. $(P \supset P) \supset X$
6. $(X \supset Q) \supset X$
7. $X \supset (Q \supset X)$
8. $(P \bullet X) \supset Y$
9. $[P \supset (Q \supset P)] \supset Y$
*10. $(Q \supset Q) \supset (A \supset X)$
11. $(P \supset X) \supset (X \supset P)$
12. $(P \supset A) \supset (B \supset X)$
13. $(X \supset P) \supset (B \supset Y)$
14. $[(P \supset B) \supset B] \supset B$
*15. $[(X \supset Q) \supset Q] \supset Q$
16. $(P \supset X) \supset ({\sim}X \supset {\sim}P)$
17. $(X \supset P) \supset ({\sim}X \supset Y)$
18. $(P \supset A) \supset (A \supset {\sim}B)$
19. $(P \supset Q) \supset (P \supset Q)$
*20. $(P \supset {\sim}{\sim}P) \supset (A \supset {\sim}B)$
21. ${\sim}(A \bullet P) \supset ({\sim}A \vee {\sim}P)$
22. ${\sim}(P \bullet X) \supset {\sim}(P \vee {\sim}X)$
23. ${\sim}(X \vee Q) \supset ({\sim}X \bullet {\sim}Q)$
24. $[P \supset (A \vee X)] \supset [(P \supset A) \supset X]$
*25. $[Q \vee (B \bullet Y)] \supset [(Q \vee B) \bullet (Q \vee Y)]$

C. Symbolize the following, using capital letters to abbreviate the simple statements involved.

*1. If Argentina mobilizes then if Brazil protests to the UN then Chile will call for a meeting of all the Latin American states.
2. If Argentina mobilizes then either Brazil will protest to the UN or Chile will call for a meeting of all the Latin American states.
3. If Argentina mobilizes then Brazil will protest to the UN and Chile will call for a meeting of all the Latin American states.
4. If Argentina mobilizes then Brazil will protest to the UN, and Chile will call for a meeting of all the Latin American states.
*5. If Argentina mobilizes and Brazil protests to the UN then Chile will call for a meeting of all the Latin American states.
6. If either Argentina mobilizes or Brazil protests to the UN then Chile will call for a meeting of all the Latin American states.
7. Either Argentina will mobilize or if Brazil protests to the UN then Chile will call for a meeting of all the Latin American states.
8. If Argentina does not mobilize then either Brazil will not protest to the UN or Chile will not call for a meeting of all the Latin American states.

9. If Argentina does not mobilize then neither will Brazil protest to the UN nor will Chile call for a meeting of all the Latin American states.

***10.** It is not the case that if Argentina mobilizes then both Brazil will protest to the UN and Chile will call for a meeting of all the Latin American states.

11. If it is not the case that Argentina mobilizes then Brazil will not protest to the UN, and Chile will call for a meeting of all the Latin American states.

12. Brazil will protest to the UN if Argentina mobilizes.

13. Brazil will protest to the UN only if Argentina mobilizes.

14. Chile will call for a meeting of all the Latin American states only if both Argentina mobilizes and Brazil protests to the UN.

***15.** Brazil will protest to the UN only if either Argentina mobilizes or Chile calls for a meeting of all the Latin American states.

16. Argentina will mobilize if either Brazil protests to the UN or Chile calls for a meeting of all the Latin American states.

17. Brazil will protest to the UN unless Chile calls for a meeting of all the Latin American states.

18. If Argentina mobilizes, then Brazil will protest to the UN unless Chile calls for a meeting of all the Latin American states.

19. Brazil will not protest to the UN unless Argentina mobilizes.

***20.** Unless Chile calls for a meeting of all the Latin American states, Brazil will protest to the UN.

21. Argentina's mobilizing is a sufficient condition for Brazil to protest to the UN.

22. Argentina's mobilizing is a necessary condition for Chile to call for a meeting of all the Latin American states.

23. If Argentina mobilizes and Brazil protests to the UN, then both Chile and the Dominican Republic will call for a meeting of all the Latin American states.

24. If Argentina mobilizes and Brazil protests to the UN, then either Chile or the Dominican Republic will call for a meeting of all the Latin American states.

***25.** If neither Chile nor the Dominican Republic calls for a meeting of all the Latin American states, then Brazil will not protest to the UN unless Argentina mobilizes.