Project 4.2
Computer Algebra Solution of Systems

Computer algebra systems can be used to solve systems as well as single differential equations. The sections below illustrate the use of *Maple*, *Mathematica*, and MATLAB to solve symbolically the first-order system

$$x' = 4x - 3y, \qquad y' = 6x - 7y \tag{1}$$

of Example 1 and the second-order mass-spring system

$$x'' + 3x - y = 0, \qquad y'' - 2x + 2y = 0 \tag{2}$$

of Example 3 in Section 4.2 of the text.

Use these examples as models for the computer algebra solution of several of the systems in Problems 1-20 and 39-46 in Section 4.2 (supplying initial conditions for the latter ones if you like). In each case, you should verify that the general solution obtained by computer is equivalent to the one obtained manually. Frequently the two presumably equivalent solutions will look different at first glance, and you will then need to explore the relation between the arbitrary constants in the computer algebra solution and the arbitrary constants in the manual solution.

Using *Maple*

To solve the system in (1) we need only define the differential equations

```
deq1 :=  diff(x(t),t) = 4*x(t) - 3*y(t);
deq2 :=  diff(y(t),t) = 6*x(t) - 7*y(t);
```

$$deq1 := \frac{\partial}{\partial x}x(t) = 4x(t) - 3y(t)$$
$$deq2 := \frac{\partial}{\partial x}y(t) = 6x(t) - 7y(t)$$

Then the command

```
dsolve( {deq1,deq2}, {x(t),y(t)} );
```

$$y(t) = \frac{9}{7}\_C1e^{(-5t)} - \frac{2}{7}\_C1e^{(2t)} + \frac{6}{7}\_C2\,e^{(2t)} - \frac{6}{7}\_C2\,e^{(-5t)},$$
$$x(t) = -\frac{3}{7}\_C1e^{(2t)} + \frac{3}{7}\_C1e^{(-5t)} - \frac{2}{7}\_C2\,e^{(-5t)} + \frac{9}{7}\_C2\,e^{(2t)}\}$$

yields (after a bit of simplification) the general solution

$$x(t) = \tfrac{1}{7}(3c_1 - c_2)e^{-5t} + \tfrac{1}{7}(-3c_1 + 9c_2)e^{2t}$$
$$y(t) = \tfrac{1}{7}(9c_1 - 6c_2)e^{-5t} + \tfrac{1}{7}(-2c_1 + 6c_2)e^{2t}. \qquad (3)$$

Is it clear that this result agrees with the general solution

$$x(t) \;=\; \tfrac{3}{2}b_1e^{2t} + \tfrac{1}{3}b_2e^{-5t}, \quad y(t) \;=\; b_1e^{2t} + b_2e^{-5t} \qquad (4)$$

found in the text? What is the relation between the constants c_1, c_2 in (3) and the constants b_1, b_2 in (4)?

To find a particular solution we need only include the desired initial conditions in the **dsolve** command. Thus

```
dsolve({deq1,deq2, x(0)=2,y(0)=-1}, {x(t),y(t)});
```

$$y(t) = -3\,e^{(-5t)} + 2\,e^{(2t)},\;\; x(t) = 3\,e^{(2t)} - e^{(-5t)}\}$$

solves the initial value problem of Example 1 in the text.

For the second-order mass-spring system in (2) we define the differential equations

```
deq3 := diff(x(t),t,t) + 3*x(t) -   y(t) = 0:
deq4 := diff(y(t),t,t) - 2*x(t) + 2*y(t) = 0:
```

and proceed to solve for the general solution.

```
dsolve( {deq3,deq4}, {x(t),y(t)} );
```

$$y(t) \;=\; \frac{2}{3}\_C1\cos(t) + \frac{1}{3}\_C1\cos(2\,t) + \frac{1}{6}\_C2\sin(2\,t) + \frac{2}{3}\_C2\sin(t)$$
$$-\frac{2}{3}\_C3\cos(2\,t) + \frac{2}{3}\_C3\cos(t) - \frac{1}{3}\_C4\sin(2\,t) + \frac{2}{3}\_C4\sin(t),$$
$$x(t) \;=\; -\frac{1}{3}\_C1\cos(2\,t) + \frac{1}{3}\_C1\cos(t) - \frac{1}{6}\_C2\sin(2\,t) + \frac{1}{3}\_C2\sin(t)$$
$$+\frac{1}{3}\_C3\cos(t) + \frac{2}{3}\_C3\cos(2\,t) + \frac{1}{3}\_C4\sin(2\,t) + \frac{1}{3}\_C4\sin(t)\}$$

Upon comparing the coefficients of $\cos(x)$, $\sin(x)$, $\cos(2x)$, and $\sin(2x)$ in these expressions for x and y, we see that this general solution is a linear combination of

- an oscillation with frequency 1 in which the two masses move synchronously with the amplitude of the second mass motion being twice the amplitude of the first, and

- an oscillation of frequency 2 in which the two masses move in opposite directions with the same amplitude of motion.

Hence it is equivalent to the general solution found in Example 3 of the text.

The two oscillations described above are more readily visible in the solution of the initial value problem

```
dsolve( {deq3,deq4, x(0)=0, y(0)=0,
          D(x)(0)=6, D(y)(0)=6}, {x(t),y(t)} );
```

$$x(t) \;=\; 4\sin t + \sin 2t,\;\; y(t) \;=\; 8\sin t - \sin 2t\}$$

Using *Mathematica*

To solve the system in (1) we need only define the differential equations

```
deq1 =    x'[t] == 4 x[t] - 3 y[t];
deq2 =    y'[t] == 6 x[t] - 7 y[t];
```

Then the command

```
DSolve[ {deq1,deq2}, {x[t],y[t]}, t ]
```

$$\left\{\left\{x(t) \to e^{-5t}\,c_1 + 3e^{2t}\,c_2,\; y(t) \to 3\,e^{-5t}\,c_1 + 2\,e^{2t}\,c_2\right\}\right\}$$

yields the general solution

$$x(t) \;=\; 3c_2e^{2t} + c_1e^{-5t}, \qquad y(t) \;=\; 2c_2e^{2t} + 3c_1e^{-5t} \tag{5}$$

Compare this result with the general solution

$$x(t) \;=\; \tfrac{3}{2}b_1e^{2t} + \tfrac{1}{3}b_2e^{-5t}, \quad y(t) \;=\; b_1e^{2t} + b_2e^{-5t} \tag{6}$$

found in the text? What is the relation between the constants c_1, c_2 in (5) and the constants b_1, b_2 in (6)?

To find a particular solution we need only include the desired initial conditions in the **DSolve** command. Thus

```
DSolve[{deq1,deq2,x[0]==2,y[0]==-1}, {x[t],y[t]}, t];
```

$$\left\{\left\{x(t) \to e^{-5t}\left(-1 + 3\,e^{7t}\right), y(t) \to e^{-5t}\left(-3 + 2\,e^{7t}\right)\right\}\right\}$$

solves the initial value problem of Example 1 in the text.

For the second-order mass-spring system in (2) we define the differential equations

```
deq3 =  x''[t] + 3*x[t] -   y[t] == 0;
deq4 =  y''[t] - 2*x[t] + 2*y[t] == 0;
```

The command

```
genSoln = DSolve[ {deq3,deq4}, {x[t],y[t]}, t ];
```

then yields $x(t)$ and $y(t)$ as rather miserable-looking linear combinations of the complex exponentials e^{it}, e^{-it}, e^{2it}, and e^{-2it} (try it for yourself and see). But we can convert these complex exponentials to trigonometric expressions by means of the commands

```
xg = First[x[t] /. genSoln] // ComplexExpand;
Collect[xg, {Cos[t],Sin[t],Cos[2t],Sin[2t]} ]
```

$$(i\,c_1 - i\,c_2)\cos(t) + (i\,c_4 - i\,c_3)\cos(2t) + (c_1 + c_2)\sin(t) + (-c_3 - c_4)\sin(2t)$$

```
yg = First[y[t] /. genSoln] // ComplexExpand;
Collect[yg, {Cos[t],Sin[t],Cos[2t],Sin[2t]} ]
```

$$(2i\,c_1 - 2i\,c_2)\cos(t) + (i\,c_3 - i\,c_4)\cos(2t) + (2c_1 + 2c_2)\sin(t) + (c_3 + c_4)\sin(2t)$$

Upon comparing the coefficients of $\cos(x)$, $\sin(x)$, $\cos(2x)$, and $\sin(2x)$ in these expressions for $x(t)$ and $y(t)$, we see that this general solution is a linear combination of

- an oscillation with frequency 1 in which the two masses move synchronously with the amplitude of the second mass motion being twice the amplitude of the first, and

- an oscillation of frequency 2 in which the two masses move in opposite directions with the same amplitude of motion.

Hence it is equivalent to the general solution found in Example 3 of the text.

The two oscillations described above are more readily visible in the solution of the initial value problem

```
partSoln =
DSolve[{deq3, deq4, x[0]==0,y[0]==0,
        x'[0]==6,y'[0]==6}, {x[t], y[t]}, t];

xp = First[x[t] /. partSoln] //
           ComplexExpand // Simplify
```

$$4\sin(t) + \sin(2t)$$

```
yp = First[y[t] /. partSoln] //
           ComplexExpand // Simplify
```

$8\sin(t) - \sin(2t)$

Using MATLAB

To solve the system in (1) we need only define the differential equations

```
deq1 =  'Dx = 4*x - 3*y';
deq2 =  'Dy = 6*x - 7*y';
```

Then the command

```
[x,y] = dsolve(deq1,deq2)
x =
  -2/7*C1*exp(-5*t)+9/7*C1*exp(2*t)-
   3/7*C2*exp(2*t)+3/7*C2*exp(-5*t)
y =
   6/7*C1*exp(2*t)-6/7*C1*exp(-5*t)+
   9/7*C2*exp(-5*t)-2/7*C2*exp(2*t)
```

yields (after a bit of simplification) the general solution

$$\begin{aligned} x(t) &= \tfrac{1}{7}(3c_1 - 2c_2)e^{-5t} + \tfrac{1}{7}(-3c_1 + 9c_2)e^{2t} \\ y(t) &= \tfrac{1}{7}(9c_1 - 6c_2)e^{-5t} + \tfrac{1}{7}(-2c_1 + 6c_2)e^{2t}. \end{aligned} \tag{7}$$

Is it clear that this result agrees with the general solution

$$x(t) \;=\; \tfrac{3}{2}b_1e^{2t} + \tfrac{1}{3}b_2e^{-5t}, \quad y(t) \;=\; b_1e^{2t} + b_2e^{-5t} \tag{8}$$

found in the text? What is the relation between the constants c_1, c_2 in (7) and the constants b_1, b_2 in (8)?

To find a particular solution we need only include the desired initial conditions in the **`dsolve`** command. Thus

```
[x,y] = dsolve(deq1,deq2,'x(0)=2','y(0)=-1')
x =
  -exp(-5*t)+3*exp(2*t)
y =
   2*exp(2*t)-3*exp(-5*t)
```

solves the initial value problem of Example 1 in the text.

For the second-order mass-spring system in (2) we define the differential equations

```
deq3 = 'D2x + 3*x - y = 0';
deq4 = 'D2y - 2*x + 2*y = 0';
```

and proceed to solve for the general solution.

```
[x,y] = dsolve(deq3,deq4)
x =
   1/3*C1*cos(t)+2/3*C1*cos(2*t)+1/3*C2*sin(2*t)
  +1/3*C2*sin(t)-1/3*C3*cos(2*t)+1/3*C3*cos(t)
  -1/6*C4*sin(2*t)+1/3*C4*sin(t)
y =
  -2/3*C1*cos(2*t)+2/3*C1*cos(t)-1/3*C2*sin(2*t)
  +2/3*C2*sin(t)+2/3*C3*cos(t)+1/3*C3*cos(2*t)
  +1/6*C4*sin(2*t)+2/3*C4*sin(t)
```

Upon comparing the coefficients of $\cos(x)$, $\sin(x)$, $\cos(2x)$, and $\sin(2x)$ in these expressions for x and y, we see that this general solution is a linear combination of

- an oscillation with frequency 1 in which the two masses move synchronously with the amplitude of the second mass motion being twice the amplitude of the first, and

- an oscillation of frequency 2 in which the two masses move in opposite directions with the same amplitude of motion.

Hence it is equivalent to the general solution found in Example 3 of the text.

The two oscillations described above are more readily visible in the solution of the initial value problem

```
[x,y] = dsolve(deq3,deq4, 'x(0)=0, y(0)=0,
               Dx(0)=6, Dy(0)=6')
x =
   sin(2*t)+4*sin(t)
y =
  -sin(2*t)+8*sin(t)
```