Mini-Excursion 3:

The Time Value of Money

Annuities and Loans

Chapter 10 introduced us to three basic models of population growth (*linear*, *exponential*, and *logistic*), and we saw that these models are applicable to the study of things other than just biological populations. The purpose of this mini-excursion is to discuss in greater detail some of the ideas behind the exponential growth model as they apply specifically to one very important population—your money! You won't find any hot stock tips or get-rich-quick real estate schemes here, but you will gain a better understanding of one of the most important principles of finance—that the value of money is time dependent, and that a dollar in your hands today is worth more than the promise of a dollar tomorrow.

Money has a **present value** and a **future value**. Most of the time, if you give up the right to x dollars today (present value) for a promise of getting the money at some future date, you should expect to get, in return for this sacrifice, something more than x dollars (future value). And of course, the same principle also works in reverse—if you are the one getting the x dollars today (either in cash or in goods), you would expect to have to pay back more than x dollars tomorrow.

The difference between the present value and the future value of money is the price that one party is paying for the *risk* that another party is taking. Every promise of future payment carries some element of risk—the risk that the promise will not be kept, and sometimes the risk that the receiving party may not be around to collect. This is a simple-sounding idea, but quantifying risk in the form of dollars and cents involves many variables and is far from trivial. But that is the business of bankers and insurance folk.

Our task here is considerably less ambitious. In this mini excursion we will explore the relationship between the present and future values of money when dealing with *ordinary annuities* and *installment loans*. The only mathematical tools we will be using are the *general compounding formula* and the *geometric sum formula*, both introduced in Chapter 10. For the reader's benefit, here are the formulas revisited.

General Compounding Formula

If $\$a$ is invested today, the future value of $\$a$ after t years is $\$a \cdot r^{t \cdot k}$ where k is the number of compounding periods per year and $r = 1 + p$ [p is the periodic interest rate expressed as a decimal (annual interest rate divided by k)].

Geometric Sum Formula

$$a + ar + ar^2 + \cdots + ar^{N-1} = \frac{a(r^N - 1)}{r - 1}$$

(The geometric sum formula does not apply when $r = 1$.)

Fixed Annuities

Before defining an annuity, we will illustrate the concept with a couple of examples.

EXAMPLE 3.1 The Lottery Winner's Dilemma

People who win a major lottery prize are immediately faced with an important financial decision—take the money in payments spread over an extended period of time (usually 25 years) or take a smaller lump sum payment up front.

Imagine you win a major lottery prize. The choice is \$350,000 cash today or \$30,000 a year paid over 25 years. Which is a better choice? Tough question, but we will be able to come up with an answer by the end of this mini-excursion.

Example 3.1 illustrates the following classic problem: How does a *present value* of $\$PV$ (the \$350,000) compare with a future value of N annual payments of $\$PMT$ (the \$30,000)? As a purely mathematical question the answer depends on PV, N, PMT plus one additional variable—the estimated risk (expressed in the form of an interest rate i) of choosing the future value option. (Note that there are also psychological and sociological aspects to this decision. If you take the lump sum and you have no self-control, you may squander the money in a short time and be miserable later. There is the "mooch factor"—all those friends, relatives, and financial advisors that will want a piece of your action, especially when you take the lump sum option. There is the question of your current financial situation and how much debt you are carrying. And on and on and on. The list is long.)

The next example is a variation on the theme first raised in Example 10.17 in Chapter 10.

EXAMPLE 3.2 Setting Up a College Trust Fund: Part 1

A mother decides to set up a college trust fund for her newborn child by making equal monthly payments into the trust fund over a period of 18 years (216 total payments). The trust fund guarantees a fixed annual interest rate of 6% compounded monthly, which equates to a periodic interest rate $p = 0.005$ (one-half of a percent). Mom's original plan was to make monthly payments of $100 at the beginning of each month. In this case, at the end of 18 years the total in the trust fund is $38,929. (This total was computed in Example 10.17, but we will go over the details again in the next example.)

Mom's problem is that the $100 monthly payments are leaving her far short of her goal of a $50,000 trust fund at the end of 18 years. What are the monthly payments she should make to the trust fund to reach her $50,000 target in 18 years? We will answer this question in the next section as well. «

Examples 3.1 and 3.2 illustrate the concept of a **fixed annuity**—a sequence of *equal* payments made or received at the end of *equal* time periods. Annuities (often disguised under different names) are so common in today's financial world that there is a good chance you may be currently involved in one or more annuities and not even realize it. You may be *making* regular deposits to save for an expensive item such as a vacation, a wedding, or college, or making regular payments on a car loan or on your credit card debt (ugh!). You could also be at the receiving end of an annuity, *getting* regular payments from an inheritance, a college trust fund set up on your behalf, or a lottery win.

The word *annuity* comes from the Latin *annua*. Ancient Roman contracts called *annua* were sold to individuals in exchange for lifetime payments made once a year. The United Kingdom started the first group annuity called the State of Tontine in 1693 in order to raise money for war. In this annuity people could buy a share of the Tontine for a fixed sum in return for annual payments for the remainder of that person's life. In the United States annuities began to be offered during the Great Depression and have since become an integral part of the modern financial world that sooner or later we all have to face.

For the remainder of this mini-excursion we will focus on two basic types of annuities: *fixed deferred annuities* and *fixed immediate annuities*. A **fixed deferred annuity** is an annuity in which a series of regular payments are made in order to produce a lump sum at a later date; a **fixed immediate annuity** is an annuity in which a lump sum is paid to generate a series of regular payments later. You can think of a deferred annuity as the process of creating a retirement nest egg (the accumulation phase) and an immediate annuity as the process of taking money out of a retirement nest egg (the payout phase).

Deferred Annuities

To measure how good a deferred annuity is we must look at its future value. The **future value** is the sum of all of the payments plus the interest earned. The college trust fund discussed in Example 3.2 is a classic example of a deferred annuity. Let's look at the example in greater detail.

EXAMPLE 3.3 Setting Up a College Trust Fund: Part 2

In Example 3.2 we mentioned that if $100 is deposited at the beginning of each month in a trust fund that pays 6% annual interest compounded monthly, the future value of the trust fund after 18 years is $38,929. Let's consider in more detail how this number comes about.

The periodic interest rate is $p = 0.06/12 = 0.005$ (an annual interest rate of 6% compounded monthly). From the general compounding formula we have that the first deposit of \$100, compounded over 216 periods (18 years = 216 months), has a future value of $\$100(1.005)^{216}$. The second deposit of \$100, compounded over 215 periods, has a future value of $\$100(1.005)^{215}$. The third deposit of \$100, compounded over 214 periods, has a future value of $\$100(1.005)^{214}$. And so on. The last deposit of \$100 is compounded over only one period and has a future value of \$100(1.005).

The future value of a deferred annuity (let's call it *FV*) is the sum of the future values of all the deposits. In this case,

$$FV = \$100(1.005)^{216} + \$100(1.005)^{215} + \cdots + \$100(1.005)^{2} + \$100(1.005)$$

The preceding sum looks like a good candidate for the geometric sum formula. To best see how the formula applies, we reverse the order of the terms and rewrite 100(1.005) as 100.5. When we do that we get

$$FV = \$100.5 + \$100.5(1.005) + \cdots + \$100.5(1.005)^{214} + \$100.5(1.005)^{215}$$

Now we can let $a = 100.50$, $r = 1.005$, and $N = 216$. Using the geometric sum formula, we get

$$FV = \$\frac{(100.50)[(1.005)^{216} - 1]}{1.005 - 1} \approx \$38{,}929$$

Before we go on, let's deconstruct the preceding expression for *FV*. The first factor in the numerator is the initial term in the sum, in this case $a = \$100.50 = \$100 \cdot (1.005)$. It represents the money in the trust fund at the end of the first month, in this case the initial monthly payment of \$100 plus \$0.50 interest for the month. (Note that when the payments are made at the *end* of the month, then the initial term a of the sum is equal to the monthly payment.) The second factor in the numerator (inside the square brackets) represents the expression $[r^N - 1]$, and the denominator represents the expression $r - 1$ (which happens to equal p).

The mother's original goal was to have the future value of the trust fund be around \$50,000. To reach this goal, the mother can either increase the length of time over which she makes the \$100 monthly payments, or, alternatively, increase the amount of the monthly payments. Let's consider both options.

Option 1: What would happen if the monthly payments to the trust fund were to stay at \$100 but the payments were extended to 19 years? To calculate the future value of this deferred annuity, all we have to do is increase the exponent N in the previous expression for *FV* from 216 to 228. Under this option the future value of the trust fund (rounded to the nearest dollar) becomes

$$FV = \$\frac{(100.5)[(1.005)^{228} - 1]}{1.005 - 1} \approx \$42{,}570$$

This future value is still far short of the \$50,000 goal, and extending the payments over more years doesn't make much sense (it is, after all, a trust fund to help with college expenses). This is clearly not the way to go.

Option 2: Suppose the monthly payments are increased to \$150 and are still made at the beginning of each month for 18 years. The only number that changes now in the original expression for *FV* is the first factor in the numerator, which becomes $150(1.005) = 150.75$. Under this option the future value of the trust fund (rounded to the nearest dollar) becomes

$$FV = \$\frac{(150.75)[(1.005)^{216} - 1]}{1.005 - 1} \approx \$58{,}393$$

This future value overshoots the \$50,000 target by quite a bit, so we need to reconsider the monthly payments. What is the correct monthly payment that ensures that the deferred annuity has a future value of \$50,000 after 18 years? Let's temporarily call this monthly payment x. Using exactly the same argument we used previously, we set up the following equation:

$$FV = \$\frac{x(1.005)[(1.005)^{216} - 1]}{1.005 - 1} = \$50{,}000$$

Solving the preceding equation for x gives

$$x = \$50{,}000 \cdot \frac{(1.005 - 1)}{(1.005)[(1.005)^{216} - 1]} \approx \$128.44$$

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When we look at it the right way, the formula for the future value of an ordinary deferred annuity matches exactly the geometric sum formula.

Future Value of a Fixed Deferred Annuity

$$FV = a \cdot \frac{r^N - 1}{r - 1}$$

To get this perfect match, we let $r = 1 + p$, where p is the periodic interest rate expressed as a percent, N be the number of equal payments made to the annuity, and a be the money in the annuity at the end of the first period. [When the payments of \$$PMT$ are made at the *end* of each period, then $a = PMT$; when the payments are made at the *start* of each period (as in Example 3.3), then $a = PMT \cdot r$.]

The formula for the future value of a fixed deferred annuity can be used to compute the size of the periodic payment necessary to reach a specific future value target over a given number of payments N. All we have to do is solve the future value formula for the unknown a:

$$a = FV \cdot \frac{r - 1}{r^N - 1}$$

If the payments are made at the end of each period, then we have $PMT = a$; if the payments are made at the beginning of each period, we have $PMT = a/r$.

Immediate Annuities

The flip side of a deferred annuity is an *immediate annuity*. The classic example of an immediate annuity is an *installment loan*, where we get a sum of money now (the present value of the loan) and pay it off in installments. When the installment payments are fixed and made over regular periods (monthly, bimonthly, etc.), we have a *fixed immediate annuity*.

There are other types of immediate annuities that are not loans—lottery winners who choose a lump sum up front rather than yearly payments are an example—but making payments on installment loans is clearly something that most of us can relate to.

Once again, the geometric sum formula will be our main mathematical tool. We will start with a very simple example to illustrate the key concept of this section—the *present value* of a loan.

EXAMPLE 3.4 Present Value of a Single Payment Loan

Jackie just turned 16 and wants to buy a car—now! Problem is, she has no money. She does have a \$6000 inheritance that she can cash in in two years, when she turns 18. She wants to borrow against that inheritance from her dad and offers to pay the entire loan back in a single payment when she turns 18. Her dad agrees to lend her the present value of the \$6000 that she will pay back in two years, and he will charge a token interest rate of 1.5% a year compounded yearly. How much money should dad spot Jackie to buy the car?

From dad's point of view, he is "investing" a sum of \$$a$ (the loan amount) to receive \$6000 in two years at an annual interest rate of $i = 0.015$. Since the interest compounds yearly, $p = i = 0.015$. Using the general compounding formula, we have

$$a(1.015)^2 = \$6000$$

and thus

$$a = \frac{\$6000}{(1.015)^2} \approx \$5824$$

In the preceding example, \$5824 is the *present value* (PV) of the \$6000 that Jackie's dad will get in two years, based on the very low interest rate of 1.5% compounded annually. The present value would be much less if Jackie's dad was dealing with a total stranger, a reflection of the fact that the interest rate (the variable that quantifies the risk factor) would be considerably higher.

In general, we can compute the present value of \$$FV$ at a time N periods into the future using the following formula. (As always, p denotes the periodic interest rate expressed as a decimal.)

Present Value of \$$FV$ at a Time N Periods in the Future

$$PV = \frac{\$FV}{(1+p)^N}$$

Note: Sometimes it is more convenient to write the formula in the alternative form $PV = \$FV \cdot (1+p)^{-N}$.

Remember that a positive exponent in a denominator becomes a negative exponent when moved to the numerator, and vice versa.

In Example 3.4 we dealt with a situation where the repayment of the loan is done in a single lump-sum payoff at the end. While this may work for small loans between family members, the typical situation is that we repay an installment loan by making a long (sometimes too long) series of payments. This makes the computation of the present value a bit more involved, but as before, the geometric sum formula will bail us out.

EXAMPLE 3.5 Present Value of an Installment Loan

You just landed a really good job and are finally able to buy that sports car you always wanted. Your local dealer is currently advertising a special offer: zero down and 72 monthly payments of \$399 a month. The annual interest rate for the financing is 9%, compounded monthly, so that the periodic interest rate is given by $p = 0.09/12 = 0.0075$. As an educated consumer you ask yourself, "How much am I paying for just the car itself, never mind the financing cost?" You can answer this question by finding the present value of the loan.

In an installment loan such as this one, each of the payments is made at a different time in the future and thus has a different present value. *The present value of the loan is the sum of the present values of the individual payments*. In this case the sum of the present values of the 72 individual payments is given by

$$PV = \$399(1.0075)^{-1} + \$399(1.0075)^{-2} + \cdots + \$399(1.0075)^{-72}.$$

The first term in the above sum is the present value of the first payment, the second term is the present value of the second payment, and so on. (For convenience, these present values are expressed using the negative exponent version of the present value formula.)

We can now apply the geometric sum formula to the preceding expression for PV. There are a couple of different ways that this can be done, but the most convenient is to choose a to equal the *smallest* term in the sum, which in this case is $\$399(1.0075)^{-72}$. Thus, $a = \$399(1.0075)^{-72}$, $r = 1.0075$, and $N = 72$. The geometric sum formula combined with a little algebraic manipulation gives:

See Exercise 7.

$$PV = \$399(1.0075)^{-72} \cdot \frac{[(1.0075)^{72} - 1]}{1.0075 - 1} = \$399 \cdot \frac{[1 - (1.0075)^{-72}]}{0.0075} \approx \$22{,}135$$

The bottom line is that you are paying $\$399 \times 72 = \$28{,}728$ to buy a \$22,135 car (the present value of the loan), and the \$6593 difference is your financing cost (i.e., the amount of interest you end up paying).

Given the periodic payment PMT, the periodic interest p, and the number of payments N, the present value of any ordinary immediate annuity can be obtained from the *present value formula*. The derivation of the formula is a direct generalization of the computations in Example 3.5.

See Exercise 8.

Present Value of a Fixed Immediate Annuity

$$PV = \$PMT \cdot \frac{[1 - (1 + p)^{-N}]}{p}$$

The root of the word *amortization* is the French word *mort*, meaning "dead" or "killed off"—as in "killing off" the loan.

When we solve the above present value formula for the periodic payment variable PMT, we get an extremely useful formula known as the *amortization formula.* The amortization formula allows us to calculate the size of the payments we need to make to pay off an installment loan with a present value of \$$PV$ given a periodic interest rate p and a number of payments N.

Amortization Formula

$$PMT = \$PV \cdot \frac{p}{[1 - (1 + p)^{-N}]}$$

We are finally in a position to make a sensible decision regarding the lottery winnings discussed in Example 3.1.

EXAMPLE 3.6 The Lottery Winner's Dilemma Revisited

Recall the dilemma facing the lottery winner of Example 3.1 (wasn't it you?): Take the present value of the lottery winnings ($350,000 in cash today) or an annuity of $30,000 a year paid over 25 years. To compare these two options we need to choose a reasonable annual rate of return on the present value (think of it as the rate of return you would expect to get if you took the $350,000 and invested it on your own).

Let's start our comparison with the assumption that we can get an annual rate of return of 5%. Using the amortization formula, we can compute the annual payments we would get on $\$PV = \$350{,}000$ when $p = 0.05$ and $N = 25$:

$$PMT = \$350{,}000 \cdot \frac{0.05}{[1 - (1.05)^{-25}]} \approx \$24{,}833$$

Clearly, under the assumption of a 5% annual interest rate we are much better off choosing the annuity of $30,000 a year paid over 25 years offered by the lottery folks. What if we assumed a higher annual rate of return—say, for example, 6%? All we have to do is change the corresponding values in the amortization formula:

$$PMT = \$350{,}000 \cdot \frac{0.06}{[1 - (1.06)^{-25}]} \approx \$27{,}379$$

We are still better off taking the annuity. It is only when we assume a rate of return of 7% that the $350,000 lump-sum payment option is a better choice than the annuity option:

$$PMT = \$350{,}000 \cdot \frac{0.07}{[1 - (1.07)^{-25}]} \approx \$30{,}034$$

A slightly different way to compare the lump-sum option and the annuity option is by computing the present value of 25 annual payments of $30,000 using the present value formula. Let's start once again assuming an annual interest rate of 5% ($p = 0.05$). Then

$$PV = \$30{,}000 \cdot \frac{[1 - (1.05)^{-25}]}{0.05} \approx \$422{,}818$$

If we raise the interest rate to 6% the present value drops to $383,501, and when the interest rate is 7% the present value is $349,607.

See Exercise 16.

The bottom line is that choosing the annuity is essentially equivalent to getting about a 7% rate of return on the present value of $350,000. Looking at it as an investment in the future, the annuity option is the safe, conservative choice. There are, however, many nonmathematical reasons why many people choose the lump-sum payment option over the annuity—control of all the money, the ability to spend as much as we want whenever we want to, and the realistic observation that we may not be around long enough to collect on the annuity.

Conclusion

Whether saving money for a retirement or getting a mortgage to buy a house, understanding the time value of money is crucial to making sound financial decisions. In this mini-excursion we saw how the *general compounding formula* and the *geometric sum formula* (both introduced in Chapter 10) can be combined and modified to give two extremely useful new formulas that allow us to compute the *future value of a fixed deferred annuity* and the *present value of a fixed immediate annuity*. An understanding of these ideas can set us free from the tyranny of bankers and loan officers and from the agony and frustration of incomprehensible finance charges

Exercises

A. The Geometric Sum Formula

1. Use the geometric sum formula to compute $\$10 + \$10(1.05) + \$10(1.05)^2 + \cdots + \$10(1.05)^{35}$.

2. Use the geometric sum formula to compute $\$500 + \$500(1.01) + \$500(1.01)^2 + \cdots + \$500(1.01)^{59}$.

3. Use the geometric sum formula to compute $\$10 + \$10(1.05)^{-1} + \$10(1.05)^{-2} + \cdots + \$10(1.05)^{-35}$.

4. Use the geometric sum formula to compute $\$500 + \$500(1.01)^{-1} + \$500(1.01)^{-2} + \cdots + \$500(1.01)^{-59}$.

5. Use the geometric sum formula to compute $\$10(1.05)^{-1} + \$10(1.05)^{-2} + \$10(1.05)^{-3} + \cdots + \$10(1.05)^{-36}$.

6. Use the geometric sum formula to compute $\$500(1.01)^{-1} + \$500(1.01)^{-2} + \cdots + \$500(1.01)^{-59} + \$500(1.01)^{-60}$.

7. Justify each of the following statements.

(a) $$\$399(1.0075)^{-1} + \$399(1.0075)^{-2} + \cdots + \$399(1.0075)^{-72} = \$399(1.0075)^{-72} \cdot \frac{[(1.0075)^{72} - 1]}{1.0075 - 1}$$

(b) $$\$399(1.0075)^{-72} \cdot \frac{[(1.0075)^{72} - 1]}{1.0075 - 1} = \$399 \cdot \frac{[1 - (1.0075)^{-72}]}{0.0075}$$

8. Justify each of the following statements.
(***Hint:*** *Try Exercise 7 first.*)

(a) $$\$PMT(1 + p)^{-1} + \$PMT(1 + p)^{-2} + \cdots + \$PMT(1 + p)^{-N} = \$PMT(1 + p)^{-N} \cdot \frac{[(1 + p)^N - 1]}{p}$$

(b) $$\$PMT(1 + p)^{-N} \cdot \frac{[(1 + p)^N - 1]}{p} = \$PMT \cdot \frac{[1 - (1 + p)^{-N}]}{p}$$

B. Annuities, Investments, and Loans

9. Starting at the age of 25, Markus invests \$2000 at the end of each year into an IRA (individual retirement account). If the IRA earns a 7.5% annual rate of return, how much money is in Markus's retirement account when he retires at the age of 65? (Assume he makes 40 annual deposits, but the last deposit does not generate any interest.)

10. Celine deposits \$400 at the end of each month into an account that returns 4.5% annual interest (compounded monthly). At the end of three years she wants to take the money in the account and use it for a 20% down payment on a new home. What is the maximum price of a home that Celine will be able to buy?

11. Donald would like to retire with a \$1 million nest egg. He plans to put money at the end of each month into an account earning 6% annual interest compounded monthly. If Donald plans to retire in 35 years, how much does he need to sock away each month?

12. Layla plans to send her daughter to Tasmania State University in 12 years. Her goal is to create a college trust fund worth \$150,000 in 12 years. If she plans to make weekly deposits into a trust fund earning 4.68% annual interest compounded weekly, how much should her weekly deposits to the trust fund be? (Assume the deposits are made at the start of each week.)

13. Freddy just remembered that he had deposited money in a savings account earning 7% annual interest at the Middletown bank 15 years ago, but he forgot exactly how much. Today he closed the account and the bank gave him a check for $1172.59. How much was Freddy's initial deposit?

14. Zero coupon bonds. A zero coupon bond is a bond that is sold now at a discount and will pay its face value at some time in the future when it matures. Suppose a zero coupon bond matures to a value of $10,000 in seven years. If the bond earns 3.5% annual interest, what is the purchase price of the bond?

15. "*CNN founder and Time Warner vice chairman Ted Turner announced Thursday night that he will donate $1 billion over the next decade to United Nations programs. The donation will be made in 10 annual installments of $100 million in Time Warner stock, he said. 'Present day value that's about $600,000, he joked.*" (***Source***: *CNN, September 19, 1997*)

Find the present value of this immediate annuity to the United Nations assuming an annual interest rate of 15.1%.

16. **(a)** Find the present value of an annuity consisting of 25 annual payments of $30,000 assuming an annual interest rate of 6%. (Assume the payments are made at the end of each year.)

(b) Find the present value of an annuity consisting of 25 annual payments of $30,000 assuming an annual interest rate of 7%. (Assume the payments are made at the end of each year.)

(c) Explain why the present value of an annuity goes down as the interest rate goes up.

17. Ned Flounders plans to donate $50 per week to his church for the next 60 years. Assuming an annual interest rate of $5\frac{1}{5}\%$ compounded weekly, find the present value of this immediate annuity to Ned's church.

18. Michael Dell, founder of Dell computers, was reputed to have a net worth of $16 billion in 2005. If Mr. Dell were to take the $16 billion and set up a 40-year immediate annuity for himself, what would his yearly payment from the annuity be? Assume a 3% annual rate of return on his money.

19. The Simpsons are planning to purchase a new home. To do so, they will need to take out a 30-year home mortgage loan of $160,000 through Middletown bank. Annual interest rates for 30-year mortgages at the Middletown bank are 5.75% compounded monthly.

(a) Compute the Simpsons' monthly mortgage payment under this loan.

(b) How much interest will the Simpsons pay over the life of the loan?

20. The Smiths are refinancing their home mortgage to a 15-year loan at 5.25% annual interest compounded monthly. Their outstanding balance on the loan is $95,000.

(a) Under their current loan, the Smiths' monthly mortgage payment is $1104. How much will the Smiths be saving in their monthly mortgage payments by refinancing? (Round your answer to the nearest dollar.)

(b) How much interest will the Smiths pay over the life of the new loan?

21. Ken just bought a house. He made a $25,000 down payment and financed the balance with a 20-year home mortgage loan with an interest rate of 5.5% compounded monthly. His monthly mortgage payment is $950. What was the selling price of the house?

C. Miscellaneous

22. You want to purchase a new car. The price of the car is $24,035. The dealer is currently offering a special promotion—you can choose a $4000 rebate or 0% financing for 72 months. Assuming you can get a 72-month car loan from your bank at an annual rate of 6% compounded monthly, which is the better deal—the 0% financing or the $4000 rebate? Justify your answer by showing your calculations.

23. Perpetuities. A **perpetuity** is a constant stream of identical payments with no end. The present value of a perpetuity of $\$C$, given an annual interest rate p (expressed as a decimal), is given by the formula

$$\$PV = \$C \cdot (1+p)^{-1} + \$C \cdot (1+p)^{-2} + \$C \cdot (1+p)^{-3} + \cdots$$

(a) Use the geometric sum formula to compute the value of the sum

$$\$C \cdot (1+p)^{-1} + \$C \cdot (1+p)^{-2} + \$C \cdot (1+p)^{-3} + \cdots + \$C \cdot (1+p)^{-N}$$

(b) Explain why as N gets larger, the value obtained in (a) gets closer and closer to $\$C/p$. (Assume $0 < p < 1$.)

24. "*Borrow $100,000 with a 6% fixed-rate mortgage and you'll pay nearly $116,000 in interest over 30 years. Put an extra $100 a month into principal payments and you'd pay just $76,000—and be done with mortgage payments nine years earlier.*" (***Source:*** *Philadelphia Inquirer finance column by Jeff Brown, November 1, 2005*)

(a) Verify that the increase in the monthly payment that is needed to pay off the mortgage in 21 years is indeed close to $100 and that roughly $40,000 will be saved in interest.

(b) How much should the monthly payment be increased in order to pay off the mortgage in 15 years? How much interest is saved in doing so?

(c) How much should the monthly payment be increased in order to pay off the mortgage in t years ($t < 30$)? Express the answer in terms of t.

25. Sam started his new job as the mathematical consultant for the XYZ Corporation on July 1, 2005. The company retirement plan works as follows: On July 1, 2006 the company deposits $1000 in Sam's retirement account, and each year thereafter, on July 1 the company deposits the amount deposited the previous year plus an additional 6%. The last deposit is made on July 1, 2035. In addition, the retirement account earns an annual interest rate of 6% compounded monthly. On July 1, 2035 all deposits and interest paid to the account stop. What is the future value of Sam's retirement account?

26. To refinance or not to refinance? When interest rates drop there are opportunities to refinance an existing home mortgage by paying the up-front expenses of refinancing and getting a mortgage at a lower interest rate. Whether it is worth doing so or not is a decision that confounds most homeowners. This exercise illustrates all the details that need to be considered to make such a decision. Suppose that your original mortgage is a 30-year loan for \$150,000 at 7.5% annual interest compounded monthly. Three years after taking out the original loan, you have an opportunity to refinance and take out a new loan at a 6% annual interest rate compounded monthly. There is an up-front refinancing cost of \$1500 (closing costs) plus 2 points (2% of the new loan).

(a) Calculate the monthly payment on the original mortgage (\$150,000 for 30 years at 7.5% annual rate compounded monthly). Call this number PMT_1. each.

(b) Calculate the outstanding balance on the original loan after making 36 monthly payments of $\$PMT_1$ each.

(c) Calculate the monthly payments if you take out a new loan on the balance computed in (b) for 27 years at a 6% annual interest rate compounded monthly). Call this number PMT_2.

(d) The monthly saving in your mortgage payment if you refinance is $MS = PMT_1 - PMT_2$. Calculate the present value of the 27 years of monthly savings of MS assuming a 3% annual rate of return. Call this number PV.

(e) Find the present value of refinancing the loan. [The present value of refinancing the loan is given by $PV - C$, where C represents the cost of refinancing (closing costs plus points).]

Projects and Papers

A. Growing Annuities

Over time the fixed payment from an annuity (such as a retirement account) will get you fewer and fewer goods and services. If prices rise 3% a year, items that cost \$1000 today will cost over \$1300 in 10 years and over \$1800 in 20 years. To combat this phenomenon, *growing* annuities in which the payments rise by a fixed percentage, say 3%, each year have been developed. In this project, you are to discuss the mathematics behind growing annuities. (See references 1 and 3.)

B. Annuities Illustrated

"*It is good educational psychology to explain difficult topics by simple diagrams. Why, then, have annuities not been put in some graphic form?*" [***Source****: J. Donald Watson (see reference 4)*]

Of 20 textbooks that Watson referenced in 1936, not one showed any type of diagram to aid in the understanding of annuities. In this project, you are to prepare a presentation illustrating the concepts discussed in this mini excursion by way of diagrams, charts, and other visual aids. While graphs and charts that show the balance on a loan or what happens to the money in a retirement account over time are useful, diagrams that illustrate the general concepts related to the time value of money are the ultimate goal of this project.

References and Further Readings

1. Kaminsky, Kenneth, *Financial Literacy: Introduction to the Mathematics of Interest, Annuities, and Insurance*. Lankham, MD: University Press of America, 2003.
2. Shapiro, David, and Thomas Streiff, *Annuities*. Chicago, IL: Dearborn Financial Publishing, 2001.
3. Taylor, Richard W., "Future Value of a Growing Annuity: A Note," *Journal of Financial Education*, Fall (1986), 17–21.
4. Watson, J. Donald, "Annuities Illustrated by Diagrams," *The Accounting Review*, Vol. 11, No. 2 (1936), 192–195.
5. Weir, David R., "Tontines, Public Finance, and Revolution in France and England, 1688–1789," *The Journal of Economic History*, Vol. 49, No. 1 (1989), 95–124.