Calculations for Time Value of Money

In this appendix, a brief explanation of the computation of the time value of money is given for readers not familiar with this subject. Modern technology has made these calculations very easy. Many computer programs have built-in time-value functions, and a large assortment of handheld calculators will solve these problems using special keys. However, some people who use these methods do not understand the rationale for the answers and merely accept the results.

At the other extreme, the calculations could be made using exponentials and/or logarithms. Such a procedure may provide a thorough learning experience, but it is tedious and time consuming. Compound interest tables have been developed to provide a relatively easy tool for solving time-value problems. They are found in Appendix A at the end of the textbook. Here we walk through four types of calculations, each representing one of the four tables.

The Excel icons marked TVM refer to the Excel spreadsheets that can be found on the companion web site.

The Future Value of a Single Sum

If you deposit $1,000 in a savings account that pays 7 percent interest annually, and you do not withdraw this interest, the original amount will keep growing. (In real life, bank interest is usually compounded more frequently than once per year, but annual compounding is assumed here. In other words, the 7 percent will be credited to the account once per year, at the end of the year.) One year later, $70 of interest will be added to the account, making the total balance $1,070. In mathematical terms, this occurrence can be written as follows: if $i$ equals the annual interest rate, then the amount of interest paid in one year equals $A$ times $i$, where $A$ is the original amount deposited. Thus, $B$, the ending balance in the account, will be

$$B = A + (A \times i) \quad \text{or} \quad B = A(1 + i)$$
In the present case, this will compute as

$$B = 1,000(1 + 0.07) = 1,000(1.07) = 1,070$$

Now, if the $1,070 is left in the account for another year, interest will be paid on the $1,070:

$$B = 1,070(1.07) = 1,144.90$$

We could also find the answer in the following way:

$$B = 1,000 \times 1.07 \times 1.07 = 1,144.90$$

This expression simplifies to 1,000 $(1.07)^2$. Thus, in general terms,

$$FV = PV(1+i)^n$$

where $FV$ = Future value

$PV$ = Present value

$i$ = Rate of interest

$n$ = Number of periods over which compounding takes place

Table A.1a in Appendix A at the end of the book presents the $(1+i)^n$ factors for a large number of periods and interest rates. Thus, if you want to find out how much your original $1,000 will grow in 8 years, you can look it up by moving down the leftmost (period) column to 8 and then moving to the right until you hit the 7 percent column. You will obtain an answer of 1.7182. Substituting into the general formula,

$$FV = 1,000(1.7182) = 1,718.20$$

This table can also be used if you know the beginning and ending amounts and want to find the rate of interest it took to go from the first to the last amount. The formula for future value can be easily transformed to solve for the interest rate. If we refer to the number that appears in the table as “factor,” then

$$FV = PV \times \text{Factor}$$

$$\text{Factor} = \frac{FV}{PV}$$

Assume you are saving for a particular purpose. You put aside $1,000 today and want to have $2,000 5 years from now:

$$\text{Factor} = \frac{2,000}{1,000} = 2$$

To find the answer, enter the table at five periods and move to the right until the number nearest to 2 is reached. This happens at 15 percent, where the $(1+i)^n$ factor equals 2.0114. Thus, you would have to earn approximately 15 percent interest to double your amount in 5 years.\(^1\)

This type of calculation is employed frequently to obtain compound growth rates, a very popular concept in business.

\(^1\)If more exact results are needed, linear interpolation should be used. We do not explain interpolation here. An explanation can be found in any basic mathematics text.
The Future Value of an Annuity

In the previous section, we deal with the compounding of a single sum. But suppose a uniform amount is set aside each period (e.g., each year), and we want to know how much will be in the account after several years.

For example, suppose five annual deposits of $500 each will be made to an account paying 7 percent annually, starting 1 year from now. What will be the amount at the end of 5 years? A simple diagram will illustrate:

The first $500 will collect interest for 4 years, the second for 3 years, and so on. This problem could be solved by making all the separate calculations and adding the items:

$$FV_a = A (1 + i)^{n-1} + A (1 + i)^{n-2} + \ldots + A (1 + i)^0$$

where $FV_a = $ Future value of the annuity

$A = $ Annuity

This expression simplifies to

$$\frac{(1 + i)^n - 1}{i}$$

Again, a table has been constructed to ease the effort involved in this calculation. Table A.1b in Appendix A shows the sum of an annuity. For the question posed here the answer is:

$$FV_a = A \text{(Factor)}$$

$$= 500 \times 5.7507$$

$$= 2,875.35$$

where “factor” is found in Table A.1b.

The preceding calculation solves for the future value of the annuity. But suppose we have a problem stated in the following form:

To finance the college education of a just-born child, the parents expect they will need $200,000 18 years from now. They believe that they can earn 7 percent on their savings. How much should they put aside each year? The future-value formula can be transposed to solve for the annuity, $A$:

$$A = \frac{FV_a}{\text{Factor}}$$

$$= \frac{200,000}{33.999}$$

$$= 5,883$$

The parents will have to deposit $5,883 per year to have $200,000 in 18 years.

Table A.1b presents factors for ordinary annuities, which means that the first payment is made at the end of the first period and the last payment occurs on the final date. But suppose the payments are to start right now, and the last payment will occur at the beginning of the last period. Such a series of payments is usually referred to as an "annuity in advance." A table has been constructed to show the sum of an annuity in advance. Table A.1c in Appendix A shows the sum of an annuity in advance.
to as an *annuity due*, and the diagram for such an arrangement, using data from the first example of this section, is:

![Diagram](image)

Table A.1b can still be applied, with one small change. The ordinary annuity factor in Table A.1b must be multiplied by \((1 + i)\). With interest again at 7 percent, the "annuity due" factor becomes 5.7507 times 1.07, or 6.1532. Thus the result is

\[
FV_a = 500(6.1532) = 3,077
\]

Note that the amount accumulated at the end of 5 years is considerably larger than in the first example. The reason? Each contribution has an extra year to compound.

**Present Value of a Single Sum**

Suppose you are to receive a sum of $500 3 years from now on. You want to receive the money today, and you would be willing to accept less, as an amount deposited today with interest would grow to a larger sum in 3 years. The question can be stated as follows: how much money accepted today would be equivalent to $500 three years from now? That depends, of course, on the rate of interest you earn on your money. As with our previous example, let us use a 7 percent interest rate.

Remember from our discussion of compounding a single sum that

\[
FV = PV \left(1 + \frac{i}{n}\right)^n
\]

We are now attempting to solve the opposite problem: we know the future value and want to find the present value. Therefore,

\[
PV = \frac{FV}{(1 + i)^n}
\]

Because

\[
\frac{1}{(1 + i)^n} = (1 + i)^{-n}
\]

this simple equation can be written as

\[
PV = FV (1 + i)^{-n}
\]

The number can be found in Table A.1c. We look up the 3-year factor at 7 percent and find it to be 0.8163. Thus,

\[
PV = 500 (0.8163) = 408
\]

Incidentally, the present-value factors (Table A.1c) are reciprocals of the future-value factors (Table A.1a). Thus, if you have only one of these tables available, you
can still do both future- and present-value calculations. For the preceding case, if only Table A.1a were available, the solution would be as follows:

\[
FV = PV(1 + i)^n \\
PV = \frac{FV}{(1 + i)^n} \\
= \frac{500}{1.225} = 408
\]

**Present Value of an Annuity**

Suppose instead of receiving just one amount in the future, you expect to receive a series of uniform payments annually for 4 years starting 1 year from now (an ordinary annuity), or, as an alternative, you can receive a lump sum today. The single amount that would be equivalent to the annuity again depends on the interest rate you can earn. Assume you are to receive four annual payments of $2,000. This is illustrated as follows:

The solution could be obtained by calculating the present value of each payment using Table A.1c and totaling the results. The first $2,000 would be discounted 1 year, the second 2 years, and so on. Using Table A.1d, however, avoids this time-consuming method. The factors contained in this table already include the discounting and summation of all the individual numbers. Thus, if the interest rate to be used is 8 percent, then

\[
PV_a = A \text{ (Factor)} \\
= 2,000 \times 3.3121 = 6,624
\]

where “factor” is found in Table A.1d. You would be indifferent between receiving a sum of $6,624 today and a series of four annual payments of $2,000.

The preceding calculations apply to an ordinary annuity. If the first payment is to be received today, a relatively simple adjustment must be made to the factors of Table A.1d. A relevant example of the present value of an annuity due is the case of a state lottery. You certainly have seen a banner headline such as “John P. Oliver Wins $5 Million in the Lottery.” True, Mr. Oliver will receive $5 million (before he pays his taxes), but not all at one time. State lotteries frequently pay the winners in twenty equal installments; the first payment is received today and the other nineteen are received annually starting 1 year from today. To convert the calculation to an annuity due, the ordinary annuity factor in Table A.1d must be multiplied by (1 + i). For twenty payments of $250,000 each, using an 8 percent interest rate, the 20-year factor is 9.8181 which, when multiplied by 1.08, becomes 10.6035. The calculation is as follows:

\[
PV_a = A \times (10.6035) \\
= 250,000 \times 10.6035 = 2,650,875
\]
Obviously, this amount is much less than the $5 million that was announced as Mr. Oliver’s winning ticket. Still, more than $2.5 million dollars is not that bad.

There is one more exercise that should be examined in this section. Suppose we know the present value of the annuity and the interest rate, but the annuity payment is unknown. Assume you want to borrow $20,000 to be paid back over 5 years in equal installments at an interest rate of 9 percent. What will your annual payments be? The factors in Table A.1 are again applicable, but we must reverse the formula:

\[
PV_a = A \text{ (Factor)}
\]

\[
A = \frac{PV_a}{\text{Factor}}
\]

\[
A = \frac{20,000}{3.8897} = 5,142
\]

**More Frequent Compounding**

All the preceding examples were in terms of years—annual compounding. But compounding may occur more frequently. Banks advertise that they compound interest on savings accounts quarterly, monthly, or even daily. The more frequent the compounding, the greater the effect of the time value of money.

In the first section of this appendix, compounding of a single sum is discussed. The formula was

\[
FV = PV (1 + i)^n
\]

If compounding occurs more often than \(n\)—for instance, \(m\) times \(n\)—then the compounding formula is revised as follows:

\[
FV = PV (1 + \frac{i}{m})^{mn}
\]

Suppose a deposit of $10,000 pays an annual interest rate of 8 percent compounded semiannually, and the time elapsed is 5 years. The equation would be:

\[
FV = 10,000 \times (1 + 0.08/2)^{5 \times 2} = 10,000 (1.04)^{10} = 10,000 (1.4802) = 14,802
\]

If compounding had been annual, the resulting amount would have been lower: $10,000 (1.4693) = 14,693.

In contrast, with quarterly compounding:

\[
FV = 10,000 (1.02)^{20} = 10,000 (1.4859) = 14,859
\]

The more frequent the compounding, the larger will be the future value.
One more application of present-value calculations is discussed here. A corporate or government bond pays periodic interest and then repays the face value on the maturity date. Interest payments are generally fixed for the life of the bond and are expressed as a percentage of the face value. Whether market interest rates rise or fall during the life of the bond, the periodic interest paid will not change. However, the price of the bond will change so the yield corresponds to the interest rate paid in the market for bonds of a similar risk class and length of life.2

As an example, we take a bond with a face and maturity value of $1,000 (the usual amount in which bonds are denominated). It will mature in 20 years (it was originally a 30-year bond, issued 10 years ago), and the stated interest rate is 8 percent. Thus, the annual interest payment is $80.3

Interest rates have risen recently, and today bonds with 20-year maturities yield 10 percent. But because the interest payment of $80 per year on the bond cannot be changed, the market value of the bond (the price someone would be willing to pay for it today) will have to decline. This situation is quite obvious. If bonds are now yielding 10 percent, a potential buyer of a $1,000 bond would require $100 of annual interest. The bond in our example pays $80 per year, with a final payment of $1,000 at maturity. To obtain its value when the current market yield is 10 percent, we compute the present value of a 20-year annuity of $80 at a 10 percent discount rate and add to it the present value of the maturity value, $1,000, also discounted to the present at 10 percent:

\[
P_0 = 80 \times (8.5136) + 1,000 \times (0.1486) = 681 + 149 = 830\]

At \(P_0 = 830\), representing the market price of the bond today, and a yield of 10 percent to maturity, this bond is equivalent to a bond selling today for $1,000 that pays $100 annually and will mature 20 years hence at $1,000.

Had interest rates dropped to 6 percent, the bond of our example, still paying $80 per year, would rise in price above its maturity value:

\[
P_0 = 80 \times (11.4699) + 1,000 \times (0.3118) = 918 + 312 = 1,230\]

Perpetual bonds do not exist in the United States. However, they have been issued elsewhere, for example, in the United Kingdom. Such bonds never mature, but they promise to pay a stated interest amount forever. If

\[
P_0 = \frac{I}{i}\]

then

\[
i = \frac{I}{P_0}\]

2The riskier, or less safe, a bond, the higher will be the interest that the market will require. The longer the length of life of the bond, the higher will be the required interest, under normal circumstances.

3Bond interest is usually paid semiannually, but for this example, annual payments are assumed.
and

\[ P_0 = \frac{l}{j} \]

If the bond pays $80 per year and the market rate is 8 percent, the price of the bond today will be its face value of $1,000. However, if the market rate of interest for bonds of this type rises to 10 percent, the bond paying $80 per year will be worth

\[ P_0 = \frac{80}{0.1} = 800 \]

A bond selling for $800 and paying $80 per year yields precisely 10 percent in perpetuity.

Fluctuations in bond prices will be greater for any change in the market interest rate, the longer the period to maturity. The price of a bond that matures 1 year hence will not decrease much when market interest rates rise. In contrast, the price of a perpetuity will fluctuate very significantly with changes in market interest yields.