SECTION 2.5 Transformations of Functions

Objectives

1. Recognize graphs of common functions.
2. Use vertical shifts to graph functions.
3. Use horizontal shifts to graph functions.
4. Use reflections to graph functions.
5. Use vertical stretching and shrinking to graph functions.
6. Graph functions involving a sequence of transformations.

Have you seen Terminator 2, The Mask, or The Matrix? These were among the first films to use spectacular effects in which a character or object having one shape was transformed in a fluid fashion into a quite different shape. The name for such a transformation is morphing. The effect allows a real actor to be seamlessly transformed into a computer-generated animation. The animation can be made to perform impossible feats before it is morphed back to the conventionally filmed image.

Like transformed movie images, the graph of one function can be turned into the graph of a different function. To do this, we need to rely on a function’s equation. Knowing that a graph is a transformation of a familiar graph makes graphing easier.

Graphs of Common Functions

Table 2.4 below and on page 236 gives names to six frequently encountered functions in algebra. The table shows each function’s graph and lists characteristics of the function. Study the shape of each graph and take a few minutes to verify the function’s characteristics from its graph. Knowing these graphs is essential for analyzing their transformations into more complicated graphs.

Table 2.4 Algebra’s Common Graphs

- **Constant Function**
  - Domain: \((-\infty, \infty)\)
  - Range: the single number \(c\)
  - Constant on \((-\infty, \infty)\)
  - Even function
  - Graph: \(f(x) = c\)

- **Identity Function**
  - Domain: \((-\infty, \infty)\)
  - Range: \((-\infty, \infty)\)
  - Increasing on \((-\infty, \infty)\)
  - Odd function
  - Graph: \(f(x) = x\)

- **Standard Quadratic Function**
  - Domain: \((-\infty, \infty)\)
  - Range: \([0, \infty)\)
  - Decreasing on \((-\infty, 0)\) and increasing on \((0, \infty)\)
  - Even function
  - Graph: \(f(x) = x^2\)
Use vertical shifts to graph functions.

### Table 2.4  Algebra's Common Graphs (continued)

<table>
<thead>
<tr>
<th>Standard Cubic Function</th>
<th>Square Root Function</th>
<th>Absolute Value Function</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Graph of f(x) = x^3" /></td>
<td><img src="image" alt="Graph of f(x) = \sqrt{x}" /></td>
<td>![Graph of f(x) =</td>
</tr>
<tr>
<td><strong>Domain:</strong> (−∞, ∞)</td>
<td><strong>Domain:</strong> [0, ∞)</td>
<td><strong>Domain:</strong> (−∞, ∞)</td>
</tr>
<tr>
<td><strong>Range:</strong> (−∞, ∞)</td>
<td><strong>Range:</strong> [0, ∞)</td>
<td><strong>Range:</strong> (−∞, ∞)</td>
</tr>
<tr>
<td><strong>Increasing on:</strong> (−∞, ∞)</td>
<td><strong>Increasing on:</strong> (0, ∞)</td>
<td><strong>Increasing on:</strong> (−∞, 0) and increasing on (0, ∞)</td>
</tr>
<tr>
<td><strong>Odd function</strong></td>
<td><strong>Neither even nor odd</strong></td>
<td><strong>Even function</strong></td>
</tr>
</tbody>
</table>

**Discovery**

The study of how changing a function's equation can affect its graph can be explored with a graphing utility. Use your graphing utility to verify the hand-drawn graphs as you read this section.

### Vertical Shifts

Let’s begin by looking at three graphs whose shapes are the same. Figure 2.40 shows the graphs. The black graph in the middle is the standard quadratic function, \( f(x) = x^2 \). Now, look at the blue graph on the top. The equation of this graph, \( g(x) = x^2 + 2 \), adds 2 to the right side of \( f(x) = x^2 \). What effect does this have on the graph of \( f \)? It shifts the graph vertically up by 2 units. \[ g(x) = x^2 + 2 = f(x) + 2 \]

Finally, look at the red graph on the bottom of Figure 2.40. The equation of this graph, \( h(x) = x^2 - 3 \), subtracts 3 from the right side of \( f(x) = x^2 \). What effect does this have on the graph of \( f \)? It shifts the graph vertically down by 3 units. \[ h(x) = x^2 - 3 = f(x) - 3 \]

In general, if \( c \) is positive, \( y = f(x) + c \) shifts the graph of \( f \) upward \( c \) units and \( y = f(x) - c \) shifts the graph of \( f \) downward \( c \) units. These are called **vertical shifts** of the graph of \( f \).
Vertical Shifts

Let \( f \) be a function and \( c \) a positive real number.
- The graph of \( y = f(x) + c \) is the graph of \( y = f(x) \) shifted \( c \) units vertically upward.
- The graph of \( y = f(x) - c \) is the graph of \( y = f(x) \) shifted \( c \) units vertically downward.

**EXAMPLE 1  Vertical Shift Down**

Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = |x| - 4 \).

**Solution** The graph of \( g(x) = |x| - 4 \) has the same shape as the graph of \( f(x) = |x| \). However, it is shifted down vertically 4 units. We have constructed a table showing some of the coordinates for \( f \) and \( g \). The graphs of \( f \) and \( g \) are shown in Figure 2.41.

| \( x \) | \( y = f(x) = |x| \) | \( (x, f(x)) \) | \( y = g(x) = |x| - 4 = f(x) - 4 \) | \( (x, g(x)) \) |
|--------|-------------------|--------------|-----------------|--------------|
| -2     | \(-2\) 2          | \(-2, 2\)    | \(-2 - 4 = -2\) | \(-2, -2\)  |
| -1     | \(-1\) 1          | \(-1, 1\)    | \(-1 - 4 = -3\) | \(-1, -3\)  |
| 0      | 0                 | \(0, 0\)     | \(0 - 4 = -4\)  | \(0, -3\)   |
| 1      | 1                 | \(1, 1\)     | \(1 - 4 = -3\)  | \(1, -3\)   |
| 2      | 2                 | \(2, 2\)     | \(2 - 4 = -2\)  | \(2, -2\)   |

**Check Point 1** Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = |x| + 3 \).

**Horizontal Shifts**

We return to the graph of \( f(x) = x^2 \), the standard quadratic function. In Figure 2.42 on the next page, the graph of function \( f \) is in the middle of the three graphs. Turn the page and verify this observation.
We know that positive numbers are to the right of zero on a number line and negative numbers are to the left of zero. This positive-negative orientation does not apply to horizontal shifts. A positive number causes a shift to the left and a negative number causes a shift to the right.

By contrast to the vertical shift situation, this time there are graphs to the left and to the right of the graph of \( f \). Look at the blue graph on the right. The equation of this graph, \( g(x) = (x - 3)^2 \), subtracts 3 from each value of \( x \) in the domain of \( f(x) = x^2 \). What effect does this have on the graph of \( f \)? It shifts the graph horizontally to the right by 3 units.

Now, look at the red graph on the left in Figure 2.42. The equation of this graph, \( h(x) = (x + 2)^2 \), adds 2 to each value of \( x \) in the domain of \( f(x) = x^2 \). What effect does this have on the graph of \( f \)? It shifts the graph horizontally to the left by 2 units.

In general, if \( c \) is positive, \( y = f(x + c) \) shifts the graph of \( f \) to the left \( c \) units and \( y = f(x - c) \) shifts the graph of \( f \) to the right \( c \) units. These are called horizontal shifts of the graph of \( f \).

**Study Tip**

We know that positive numbers are to the right of zero on a number line and negative numbers are to the left of zero. This positive-negative orientation does not apply to horizontal shifts. A positive number causes a shift to the left and a negative number causes a shift to the right.

**Horizontal Shifts**

Let \( f \) be a function and \( c \) a positive real number.

- The graph of \( y = f(x + c) \) is the graph of \( y = f(x) \) shifted to the left \( c \) units.
- The graph of \( y = f(x - c) \) is the graph of \( y = f(x) \) shifted to the right \( c \) units.

**EXAMPLE 2 Horizontal Shift to the Left**

Use the graph of \( f(x) = \sqrt{x} \) to obtain the graph of \( g(x) = \sqrt{x + 5} \).

**Solution** Compare the equations for \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt{x + 5} \). The equation for \( g \) adds 5 to each value of \( x \) in the domain of \( f \).

\[
y = g(x) = \sqrt{x + 5} = f(x + 5)
\]

The graph of \( g(x) = \sqrt{x + 5} \) has the same shape as the graph of \( f(x) = \sqrt{x} \). However, it is shifted horizontally to the left 5 units. We have created tables on the next page showing some of the coordinates for \( f \) and \( g \). As shown in Figure 2.43, every point in the graph of \( g \) is exactly 5 units to the left of a corresponding point on the graph of \( f \).
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Some functions can be graphed by combining horizontal and vertical shifts. These functions will be variations of a function whose equation you know how to graph, such as the standard quadratic function, the standard cubic function, the square root function, or the absolute value function.

In our next example, we will use the graph of the standard quadratic function, to obtain the graph of \( h(x) = (x + 1)^2 - 3 \). We will graph three functions:

\[
\begin{align*}
  f(x) &= x^2, \\
  g(x) &= (x + 1)^2, \\
  h(x) &= (x + 1)^2 - 3.
\end{align*}
\]

**Discovery**

Work Example 3 by first shifting the graph of \( f(x) = x^2 \) three units down, graphing \( g(x) = x^2 - 3 \). Now, shift this graph one unit left to graph \( h(x) = (x + 1)^2 - 3 \). Did you obtain the graph in Figure 2.44(c)? What can you conclude?

**EXAMPLE 3** Combining Horizontal and Vertical Shifts

Use the graph of \( f(x) = x^2 \) to obtain the graph of \( h(x) = (x + 1)^2 - 3 \).

**Solution**

**Step 1** Graph \( f(x) = x^2 \). The graph of the standard quadratic function is shown in Figure 2.44(a). We’ve identified three points on the graph.

**Step 2** Graph \( g(x) = (x + 1)^2 \). Because we add 1 to each value of \( x \) in the domain of the standard quadratic function, \( f(x) = x^2 \), we shift the graph of \( f \) horizontally one unit to the left. This is shown in Figure 2.44(b). Notice that every point in the graph in Figure 2.44(b) has an \( x \)-coordinate that is one less than the \( x \)-coordinate for the corresponding point in the graph in Figure 2.44(a).
Step 3 Graph $h(x) = (x + 1)^2 - 3$. Because we subtract 3, we shift the graph in Figure 2.44(b) vertically down 3 units. The graph is shown in Figure 2.44(c). Notice that every point in the graph in Figure 2.44(c) has a $y$-coordinate that is three less than the $y$-coordinate of the corresponding point in the graph in Figure 2.44(b).

Check Point 3 Use the graph of $f(x) = \sqrt{x}$ to obtain the graph of $h(x) = \sqrt{x - 1} - 2$.

Reflections of Graphs

This photograph shows a reflection of an old bridge in a Maryland river. This perfect reflection occurs because the surface of the water is absolutely still. A mild breeze rippling the water’s surface would distort the reflection.

Is it possible for graphs to have mirror-like qualities? Yes. Figure 2.45 shows the graphs of $f(x) = x^2$ and $g(x) = -x^2$. The graph of $g$ is a reflection about the $x$-axis of the graph of $f$. In general, the graph of $y = -f(x)$ reflects the graph of $f$ about the $x$-axis. Thus, the graph of $g$ is a reflection of the graph of $f$ about the $x$-axis because $g(x) = -x^2 = -f(x)$.

Reflection about the $x$-Axis

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the $x$-axis.
EXAMPLE 4 Reflection about the x-Axis

Use the graph of \( f(x) = \sqrt{x} \) to obtain the graph of \( g(x) = -\sqrt{x} \).

Solution Compare the equations for \( f(x) = \sqrt{x} \) and \( g(x) = -\sqrt{x} \). The graph of \( g \) is a reflection about the \( x \)-axis of the graph of \( f \) because

\[
g(x) = -\sqrt{x} = -f(x).
\]

We have created a table showing some of the coordinates for \( f \) and \( g \). The graphs of \( f \) and \( g \) are shown in Figure 2.46.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \sqrt{x} )</th>
<th>( (x, f(x)) )</th>
<th>( g(x) = -\sqrt{x} )</th>
<th>( (x, g(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sqrt{0} = 0 )</td>
<td>( (0, 0) )</td>
<td>( -\sqrt{0} = 0 )</td>
<td>( (0, 0) )</td>
</tr>
<tr>
<td>1</td>
<td>( \sqrt{1} = 1 )</td>
<td>( (1, 1) )</td>
<td>( -\sqrt{1} = -1 )</td>
<td>( (1, -1) )</td>
</tr>
<tr>
<td>4</td>
<td>( \sqrt{4} = 2 )</td>
<td>( (4, 2) )</td>
<td>( -\sqrt{4} = -2 )</td>
<td>( (4, -2) )</td>
</tr>
</tbody>
</table>

Check Point 4 Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = -|x| \).

It is also possible to reflect graphs about the \( y \)-axis.

Reflection about the \( y \)-Axis

The graph of \( y = f(-x) \) is the graph of \( y = f(x) \) reflected about the \( y \)-axis.

EXAMPLE 5 Reflection about the y-Axis

Use the graph of \( f(x) = \sqrt{x} \) to obtain the graph of \( h(x) = \sqrt{-x} \).

Solution Compare the equations for \( f(x) = \sqrt{x} \) and \( h(x) = \sqrt{-x} \). The graph of \( h \) is a reflection about the \( y \)-axis of the graph of \( f \) because

\[
h(x) = \sqrt{-x} = f(-x).
\]

We have created tables showing some of the coordinates for \( f \) and \( h \). The graphs of \( f \) and \( h \) are shown in Figure 2.47.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \sqrt{x} )</th>
<th>( (x, f(x)) )</th>
<th>( h(x) = \sqrt{-x} )</th>
<th>( (x, h(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sqrt{0} = 0 )</td>
<td>( (0, 0) )</td>
<td>( \sqrt{-0} = 0 )</td>
<td>( (0, 0) )</td>
</tr>
<tr>
<td>1</td>
<td>( \sqrt{1} = 1 )</td>
<td>( (1, 1) )</td>
<td>( \sqrt{-1} = \sqrt{-1} )</td>
<td>( (1, 1) )</td>
</tr>
<tr>
<td>4</td>
<td>( \sqrt{4} = 2 )</td>
<td>( (4, 2) )</td>
<td>( \sqrt{-4} = \sqrt{-4} )</td>
<td>( (4, -2) )</td>
</tr>
</tbody>
</table>

Check Point 5 Use the graph of \( f(x) = \sqrt{x - 1} \) in Figure 2.48 to obtain the graph of \( h(x) = \sqrt{-x - 1} \).
Use vertical stretching and shrinking to graph functions.

![Graph of f(x) = x^2, g(x) = 2x^2, and h(x) = \frac{1}{2}x^2]

**Figure 2.49** Stretching and shrinking

Vertical Stretching and Shrinking

Morphing does much more than move an image horizontally, vertically, or about an axis. An object having one shape is transformed into a different shape. Horizontal shifts, vertical shifts, and reflections do not change the basic shape of a graph. How can we shrink and stretch graphs, thereby altering their basic shapes?

Look at the three graphs in Figure 2.49. The black graph in the middle is the graph of the standard quadratic function, \( f(x) = x^2 \). Now, look at the blue graph on the top. The equation of this graph is \( g(x) = 2x^2 \). Thus, for each \( x \), the \( y \)-coordinate of \( g \) is 2 times as large as the corresponding \( y \)-coordinate on the graph of \( f \). The result is a narrower graph. We say that the graph of \( g \) is obtained by vertically stretching the graph of \( f \). Now, look at the red graph on the bottom. The equation of this graph is \( h(x) = \frac{1}{2}x^2 \), or \( h(x) = \frac{1}{2}f(x) \). Thus, for each \( x \), the \( y \)-coordinate of \( h \) is one-half as large as the corresponding \( y \)-coordinate on the graph of \( f \). The result is a wider graph. We say that the graph of \( h \) is obtained by vertically shrinking the graph of \( f \).

These observations can be summarized as follows:

**Stretching and Shrinking Graphs**

Let \( f \) be a function and \( c \) a positive real number.

- If \( c > 1 \), the graph of \( y = cf(x) \) is the graph of \( y = f(x) \) vertically stretched by multiplying each of its \( y \)-coordinates by \( c \).
- If \( 0 < c < 1 \), the graph of \( y = cf(x) \) is the graph of \( y = f(x) \) vertically shrunk by multiplying each of its \( y \)-coordinates by \( c \).

**EXAMPLE 6 Vertically Stretching a Graph**

Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = 2|x| \).

**Solution** The graph of \( g(x) = 2|x| \) is obtained by vertically stretching the graph of \( f(x) = |x| \). We have constructed a table showing some of the coordinates for \( f \) and \( g \). Observe that the \( y \)-coordinate on the graph of \( g \) is twice as large as the corresponding \( y \)-coordinate on the graph of \( f \). The graphs of \( f \) and \( g \) are shown in Figure 2.50.

| \( x \) | \( f(x) = |x| \) | \( (x, f(x)) \) | \( g(x) = 2|x| = 2f(x) \) | \( (x, g(x)) \) |
|---|---|---|---|---|
| -2 | -2 = 2 | (-2, 2) | 2-2 = 4 | (-2, 4) |
| -1 | -1 = 1 | (-1, 1) | 2-1 = 2 | (-1, 2) |
| 0 | 0 = 0 | (0, 0) | 2|0| = 0 | (0, 0) |
| 1 | 1 = 1 | (1, 1) | 2|1| = 2 | (1, 2) |
| 2 | 2 = 2 | (2, 2) | 2|2| = 4 | (2, 4) |

**Check Point** Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = 3|x| \).
Table 2.5  Summary of Transformations

In each case, \( c \) represents a positive real number.

<table>
<thead>
<tr>
<th>To Graph:</th>
<th>Draw the Graph of ( f ) and:</th>
<th>Changes in the Equation of ( y = f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical shifts</td>
<td>Raise the graph of ( f ) by ( c ) units.</td>
<td>( c ) is added to ( f(x) ).</td>
</tr>
<tr>
<td>( y = f(x) + c )</td>
<td>Lower the graph of ( f ) by ( c ) units.</td>
<td>( c ) is subtracted from ( f(x) ).</td>
</tr>
<tr>
<td>( y = f(x) - c )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horizontal shifts</td>
<td>Shift the graph of ( f ) to the left ( c ) units.</td>
<td>( x ) is replaced with ( x + c ).</td>
</tr>
<tr>
<td>( y = f(x + c) )</td>
<td>Shift the graph of ( f ) to the right ( c ) units.</td>
<td>( x ) is replaced with ( x - c ).</td>
</tr>
<tr>
<td>( y = f(x - c) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reflection about the ( x )-axis</td>
<td>Reflect the graph of ( f ) about the ( x )-axis.</td>
<td>( f(x) ) is multiplied by (-1).</td>
</tr>
<tr>
<td>( y = -f(x) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reflection about the ( y )-axis</td>
<td>Reflect the graph of ( f ) about the ( y )-axis.</td>
<td>( x ) is replaced with (-x).</td>
</tr>
<tr>
<td>( y = f(-x) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vertical stretching or shrinking</td>
<td>Multiply each ( y )-coordinate of ( y = f(x) ) by ( c ), vertically stretching the graph of ( f ).</td>
<td>( f(x) ) is multiplied by ( c ), ( c &gt; 1 ).</td>
</tr>
<tr>
<td>( y = cf(x) ), ( c &gt; 1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = cf(x) ), ( 0 &lt; c &lt; 1 )</td>
<td>Multiply each ( y )-coordinate of ( y = f(x) ) by ( c ), vertically shrinking the graph of ( f ).</td>
<td>( f(x) ) is multiplied by ( c ), ( 0 &lt; c &lt; 1 ).</td>
</tr>
</tbody>
</table>
A function involving more than one transformation can be graphed by performing transformations in the following order:

1. Horizontal shifting
2. Vertical stretching or shrinking
3. Reflecting
4. Vertical shifting

**EXAMPLE 8** Graphing Using a Sequence of Transformations

Use the graph of \( f(x) = \sqrt{x} \) to graph \( g(x) = \sqrt{1-x} + 3 \).

**Solution** The following sequence of steps is illustrated in Figure 2.52. We begin with the graph of \( f(x) = \sqrt{x} \).

**Step 1** Horizontal Shifting Graph because \( x \) is replaced with \( x+1 \), the graph of \( f(x) = \sqrt{x} \) is shifted 1 unit to the left.

**Step 2** Vertical Stretching or Shrinking Because the equation \( y = \sqrt{x+1} \) is not multiplied by a constant in no stretching or shrinking is involved.

**Step 3** Reflecting We are interested in graphing \( y = \sqrt{1-x} + 3 \), or \( y = \sqrt{-x+1} + 3 \). We have now graphed \( y = \sqrt{x+1} \). We can graph \( y = \sqrt{-x+1} \) by noting that \( x \) is replaced with \(-x\). Thus, we graph \( y = \sqrt{-x+1} \) by reflecting the graph of \( y = \sqrt{x+1} \) about the \( y \)-axis.

**Step 4** Vertical Shifting We can use the graph of \( y = \sqrt{1-x} \) to get the graph of \( g(x) = \sqrt{1-x} + 3 \). Because 3 is added, shift the graph of \( y = \sqrt{1-x} \) up 3 units.

![Figure 2.52](image-url)

**Figure 2.52** Using \( f(x) = \sqrt{x} \) to graph \( g(x) = \sqrt{1-x} + 3 \)

Use the graph of \( f(x) = x^2 \) to graph \( g(x) = -(x - 2)^2 + 3 \).

**EXERCISE SET 2.5**

**Practice Exercises**

In Exercises 1–10, begin by graphing the standard quadratic function, \( f(x) = x^2 \). Then use transformations of this graph to graph the given function.

1. \( g(x) = x^2 - 2 \)
2. \( g(x) = x^2 - 1 \)
3. \( g(x) = (x - 2)^2 \)
4. \( g(x) = (x - 1)^2 \)
5. \( h(x) = -(x - 2)^2 \)
6. \( h(x) = -(x - 1)^2 \)
7. \( h(x) = (x - 2)^2 + 1 \)
8. \( h(x) = (x - 1)^2 + 2 \)
9. \( g(x) = 2(x - 2)^2 \)
10. \( g(x) = \frac{1}{2}(x - 1)^2 \)
In Exercises 11–22, begin by graphing the square root function, \( f(x) = \sqrt{x} \). Then use transformations of this graph to graph the given function.

11. \( g(x) = \sqrt{x} + 2 \) 
12. \( g(x) = \sqrt{x} + 1 \)
13. \( g(x) = \sqrt{x} + \frac{1}{2} \) 
14. \( g(x) = \sqrt{x} + \frac{1}{1} \)
15. \( h(x) = -\sqrt{x} + 2 \) 
16. \( h(x) = -\sqrt{x} + 1 \)
17. \( h(x) = \sqrt{-x} + 2 \) 
18. \( h(x) = \sqrt{-x} + 1 \)
19. \( g(x) = \frac{1}{2}\sqrt{x} + 2 \) 
20. \( g(x) = 2\sqrt{x} + 1 \)
21. \( h(x) = \sqrt{x} + 2 - 2 \) 
22. \( h(x) = \sqrt{x} + 1 - 1 \)

In Exercises 23–34, begin by graphing the absolute value function, \( f(x) = |x| \). Then use transformations of this graph to graph the given function.

23. \( g(x) = |x| + 4 \) 
24. \( g(x) = |x| + 3 \)
25. \( g(x) = |x + 4| \) 
26. \( g(x) = |x + 3| \)
27. \( h(x) = |x + 4| - 2 \) 
28. \( h(x) = |x + 3| - 2 \)
29. \( h(x) = -|x + 4| \) 
30. \( h(x) = -|x + 3| \)
31. \( g(x) = -|x + 4| + 1 \) 
32. \( g(x) = -|x + 4| + 2 \)
33. \( h(x) = 2|x + 4| \) 
34. \( h(x) = 2|x + 3| \)

In Exercises 35–44, begin by graphing the standard cubic function, \( f(x) = x^3 \). Then use transformations of this graph to graph the given function.

35. \( g(x) = x^3 - 3 \) 
36. \( g(x) = x^3 - 2 \)
37. \( g(x) = (x - 3)^3 \) 
38. \( g(x) = (x - 2)^3 \)
39. \( h(x) = -x^3 \) 
40. \( h(x) = -(x - 2)^3 \)
41. \( h(x) = \frac{1}{2}x^3 \) 
42. \( h(x) = \frac{1}{2}x^3 \)
43. \( r(x) = (x - 3)^3 + 2 \) 
44. \( r(x) = (x - 2)^3 + 1 \)

In Exercises 45–52, use the graph of the function \( f \) to sketch the graph of the given function \( g \).

45. \( g(x) = f(x) + 1 \) 
46. \( g(x) = f(x) + 2 \)
47. \( g(x) = f(x + 1) \) 
48. \( g(x) = f(x + 2) \)
49. \( g(x) = -f(x) \) 
50. \( g(x) = \frac{1}{2}f(x) \)
51. \( g(x) = \frac{1}{2}f(x) + 1 \) 
52. \( g(x) = -f(x + 2) \)

In Exercises 53–56, write a possible equation for the function whose graph is shown. Each graph shows a transformation of a common function.

53. \([-2, 8, 1] \) by \([-1, 4, 1] \)
54. \([-3, 3, 1] \) by \([-6, 6, 1] \)
55. \([-3, 3, 1] \) by \([-5, 10, 1] \)
56. \([-1, 9, 1] \) by \([-1, 5, 1] \)
Application Exercises

57. The function \( f(x) = 2.9\sqrt{x} + 20.1 \) models the median height, \( f(x) \), in inches, of boys who are \( x \) months of age. The graph of \( f \) is shown.

\[ y \]
\[ f(x) = 2.9\sqrt{x} + 20.1 \]
\[ 0 \quad 10 \quad 20 \quad 30 \quad 40 \quad 50 \quad 60 \]
\[ x \]

\[ \text{Age (months)} \]
\[ \text{Median Height (inches)} \]

a. Describe how the graph can be obtained using transformations of the square root function \( f(x) = \sqrt{x} \).
b. According to the model, what is the median height of boys who are 48 months, or four years, old? Use a calculator and round to the nearest tenth. The actual median height for boys at 48 months is 40.8 inches. How well does the model describe the actual height?
c. Use the model to find the average rate of change, in inches per month, between birth and 10 months. Round to the nearest tenth.
d. Use the model to find the average rate of change, in inches per month, between 50 and 60 months. Round to the nearest tenth. How does this compare with your answer in part (c)? How is this difference shown by your graph?
e. Rewrite the function so that it represents the amount, \( f(x) \), in billions of dollars, of new student loans \( x \) years after 1995.

58. The graph shows the amount of money, in billions of dollars, of new student loans from 1993 through 2000.

\[ \begin{array}{ccccccccc}
\text{Loan Amount (billion)} & 12.0 & 18.0 & 22.0 & 24.0 & 25.5 & 27.2 & 28.7 & 30.0 \\
\end{array} \]

Source: U.S. Department of Education

The data shown can be modeled by the function \( f(x) = 6.75\sqrt{x} + 12 \), where \( f(x) \) is the amount, in billion of dollars, of new student loans \( x \) years after 1993.

a. Describe how the graph of \( f \) can be obtained using transformations of the square root function \( f(x) = \sqrt{x} \). Then sketch the graph of \( f \) over the interval \( 0 \leq x \leq 9 \). If applicable, use a graphing utility to verify your hand-drawn graph.
b. According to the model, how much was loaned in 2000? Round to the nearest tenth of a billion. How well does the model describe the actual data?
c. Use the model to find the average rate of change, in billions of dollars per year, between 1993 and 1995. Round to the nearest tenth.
d. Use the model to find the average rate of change, in billions of dollars per year, between 1998 and 2000. Round to the nearest tenth. How does this compare with you answer in part (c)? How is this difference shown by your graph?
e. Rewrite the function so that it represents the amount, \( f(x) \), in billions of dollars, of new student loans \( x \) years after 1995.

Writing in Mathematics

59. What must be done to a function’s equation so that its graph is shifted vertically upward?

60. What must be done to a function’s equation so that its graph is shifted horizontally to the right?

61. What must be done to a function’s equation so that its graph is reflected about the x-axis?

62. What must be done to a function’s equation so that its graph is reflected about the y-axis?

63. What must be done to a function’s equation so that its graph is stretched?

Technology Exercises

64. a. Use a graphing utility to graph \( f(x) = x^2 + 1 \).
b. Graph \( f(x) = x^2 + 1 \), \( g(x) = f(2x) \), \( h(x) = f(3x) \), and \( k(x) = f(4x) \) in the same viewing rectangle.
c. Describe the relationship among the graphs of \( f \), \( g \), \( h \), and \( k \), with emphasis on different values of \( x \) for points on all four graphs that give the same \( y \)-coordinate.
d. Generalize by describing the relationship between the graph of \( f \) and the graph of \( g \), where \( g(x) = f(cx) \) for \( c > 1 \).
e. Try out your generalization by sketching the graphs of \( f(cx) \) for \( c = 1 \), \( c = 2 \), \( c = 3 \), and \( c = 4 \) for a function of your choice.

65. a. Use a graphing utility to graph \( f(x) = x^2 + 1 \).
b. Graph \( f(x) = x^2 + 1 \), \( g(x) = f(\frac{1}{2}x) \), and \( h(x) = f(\frac{1}{3}x) \) in the same viewing rectangle.
c. Describe the relationship among the graphs of \( f \), \( g \), and \( h \), with emphasis on different values of \( x \) for points on all three graphs that give the same \( y \)-coordinate.
d. Generalize by describing the relationship between the graph of \( f \) and the graph of \( g \), where \( g(x) = f(cx) \) for \( 0 < c < 1 \).
e. Try out your generalization by sketching the graphs of \( f(cx) \) for \( c = 1 \), \( c = \frac{1}{2} \), and \( c = \frac{1}{3} \) for a function of your choice.
Critical Thinking Exercises

66. Which one of the following is true?
   a. If \( f(x) = |x| \) and \( g(x) = |x + 3| + 3 \), then the graph of \( g \) is a translation of three units to the right and three units upward of the graph of \( f \).
   b. If \( f(x) = -\sqrt{x} \) and \( g(x) = \sqrt{-x} \), then \( f \) and \( g \) have identical graphs.
   c. If \( f(x) = x^2 \) and \( g(x) = 5(x^2 - 2) \), then the graph of \( g \) can be obtained from the graph of \( f \) by stretching \( f \) five units followed by a downward shift of two units.
   d. If \( f(x) = x^3 \) and \( g(x) = -(x - 3)^3 - 4 \), then the graph of \( g \) can be obtained from the graph of \( f \) by moving \( f \) three units to the right, reflecting in the \( x \)-axis, and then moving the resulting graph down four units.

In Exercises 67–70, functions \( f \) and \( g \) are graphed in the same rectangular coordinate system. If \( g \) is obtained from \( f \) through a sequence of transformations, find an equation for \( g \).

67. 

68. 

69. 

70. 

For Exercises 71–74, assume that \((a, b)\) is a point on the graph of \( f \). What is the corresponding point on the graph of each of the following functions?

71. \( y = f(-x) \)  
72. \( y = 2f(x) \)  
73. \( y = f(x - 3) \)  
74. \( y = f(x) - 3 \)

Group Exercise

75. This activity is a group research project on morphing and should result in a presentation made by group members to the entire class. Be sure to include morphing images that will intrigue class members. You should have no problem finding an array of fascinating images online. Also include a discussion of films using spectacular morphing effects. Rent videos of these films and show appropriate excerpts.
SECTION 2.6 Combinations of Functions; Composite Functions

Objectives
1. Combine functions arithmetically, specifying domains.
2. Form composite functions.
3. Determine domains for composite functions.
4. Write functions as compositions.

They say a fool and his money are soon parted and the rest of us just wait to be taxed. It’s hard to believe that the United States was a low-tax country in the early part of the twentieth century. Figure 2.53 shows how the tax burden has grown since then. We can use the information shown to illustrate how two functions can be combined to form a new function. In this section, you will learn how to combine functions to obtain new functions.

![U.S. Per Capita Tax Burden in 2000 Dollars](image)

**Figure 2.53** Source: Tax Foundation

**Combinations of Functions**

To begin our discussion, take a look at the information shown for the year 2000. The total per capita tax burden is approximately $10,500. The per capita state and local tax is approximately $3400. The per capita federal tax is the difference between these amounts.

\[
\text{Per capita federal tax} = 10,500 - 3400 = 7100
\]
We can think of this subtraction as the subtraction of function values. We do this by introducing the following functions:

Let $T(x) =$ total per capita tax in year $x$.
Let $S(x) =$ per capita state and local tax in year $x$.

Using Figure 2.53, we see that

$$T(2000) = 10,500 \quad \text{and} \quad S(2000) = 3400.$$

We can subtract these function values by introducing a new function, $T - S$, defined by the subtraction of $T(x)$ and $S(x)$. Thus,

$$(T - S)(x) = T(x) - S(x) = \text{total per capita tax in year } x \quad \text{minus state and local per capita tax in year } x.$$

For example,


Figure 2.53 illustrates that information involving differences of functions often appears in graphs seen in newspapers and magazines. Like numbers and algebraic expressions, two functions can be added, subtracted multiplied, or divided as long as there are numbers common to the domains of both functions. The common domain for functions $T$ and $S$ in Figure 2.53 is

$$\{1900, 1901, 1902, 1903, \ldots, 2000\}.$$

Because functions are usually given as equations, we perform operations by carrying out these operations with the algebraic expressions that appear on the right side of the equations. For example, we can combine the following two functions using addition:

$$f(x) = 2x + 1 \quad \text{and} \quad g(x) = x^2 - 4.$$ 

To do so, we add the terms to the right of the equal sign for $f(x)$ to the terms to the right of the equal sign for $g(x)$. Here is how it’s done:

$$f(x) + g(x) = (2x + 1) + (x^2 - 4) \quad \text{Add terms for } f(x) \text{ and } g(x).$$

$$= 2x + 1 + x^2 - 4 \quad \text{Combine like terms.}$$

$$= x^2 + 2x - 3 \quad \text{Arrange terms in descending powers of } x.$$

The name of this new function is $f + g$. Thus, the sum $f + g$ is the function defined by $(f + g)(x) = x^2 + 2x - 3$. The domain of $f + g$ consists of the numbers $x$ that are in the domain of $f$ and in the domain of $g$. Because neither $f$ nor $g$ contains division or even roots, the domain of each function is the set of all real numbers. Thus, the domain of $f + g$ is also the set of all real numbers.
EXAMPLE 1 Finding the Sum of Two Functions

Let \( f(x) = x^2 - 3 \) and \( g(x) = 4x + 5 \). Find:

a. \( (f + g)(x) \)

b. \( (f + g)(3) \).

Solution

a. \( (f + g)(x) = f(x) + g(x) = (x^2 - 3) + (4x + 5) = x^2 + 4x + 2 \). Thus,

\[ (f + g)(x) = x^2 + 4x + 2. \]

b. We find \( (f + g)(3) \) by substituting 3 for \( x \) in the equation for \( f + g \).

\[ (f + g)(3) = 3^2 + 4 \cdot 3 + 2 = 9 + 12 + 2 = 23 \]

Check Point

Let \( f(x) = 3x^2 + 4x - 1 \) and \( g(x) = 2x + 7 \). Find:

a. \( (f + g)(x) \)

b. \( (f + g)(4) \).

Here is a general definition for function addition:

The Sum of Functions

Let \( f \) and \( g \) be two functions. The sum \( f + g \) is the function defined by

\[ (f + g)(x) = f(x) + g(x). \]

The domain of \( f + g \) is the set of all real numbers that are common to the domain of \( f \) and the domain of \( g \).

EXAMPLE 2 Adding Functions and Determining the Domain

Let \( f(x) = \sqrt{x + 3} \) and \( g(x) = \sqrt{x - 2} \). Find:

a. \( (f + g)(x) \)

b. the domain of \( f + g \).

Solution

a. \( (f + g)(x) = f(x) + g(x) = \sqrt{x + 3} + \sqrt{x - 2} \)

b. The domain of \( f + g \) is the set of all real numbers that are common to the domain of \( f \) and the domain of \( g \). Thus, we must find the domains of \( f \) and \( g \). We will do so for \( f \) first.

Note that \( f(x) = \sqrt{x + 3} \) is a function involving the square root of \( x + 3 \). Because the square root of a negative quantity is not a real number, the value of \( x + 3 \) must be nonnegative. Thus, the domain of \( f \) is all \( x \) such that \( x + 3 \geq 0 \). Equivalently, the domain is \( \{x | x \geq -3\} \), or \([-3, \infty)\).

Likewise, \( g(x) = \sqrt{x - 2} \) is also a square root function. Because the square root of a negative quantity is not a real number, the value of \( x - 2 \) must be nonnegative. Thus, the domain of \( g \) is all \( x \) such that \( x - 2 \geq 0 \). Equivalently, the domain is \( \{x | x \geq 2\} \), or \([2, \infty)\).

Now, we can use a number line to determine the domain of \( f + g \). Figure 2.54 shows the domain of \( f \) in blue and the domain of \( g \) in red. Can you see that all real numbers greater than or equal to 2 are common to both domains? This is shown in purple on the number line. Thus, the domain of \( f + g \) is \([2, \infty)\).
Let and Find:

a. b.

the domain of

We can also combine functions using subtraction, multiplication, and division by performing operations with the algebraic expressions that appear on the right side of the equations. For example, the functions and

can be combined to form the difference, product, and quotient of

Here's how it's done.

Just like the domain for

the domain for each of these functions consists of all real numbers that are common to the domains of

In the case of the quotient function we must remember not to divide by 0, so we add the further restriction that

The following definitions summarize our discussion:

**Definitions: Sum, Difference, Product, and Quotient of Functions**

Let \( f \) and \( g \) be two functions. The **sum** \( f + g \), the **difference** \( f - g \), the **product** \( fg \), and the **quotient** \( \frac{f}{g} \) are functions whose domains are the set of all real numbers common to the domains of \( f \) and \( g \), defined as follows:

1. **Sum:** \((f + g)(x) = f(x) + g(x)\)
2. **Difference:** \((f - g)(x) = f(x) - g(x)\)
3. **Product:** \((fg)(x) = f(x) \cdot g(x)\)
4. **Quotient:** \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\), provided \(g(x) \neq 0\)
EXAMPLE 3 Combining Functions

If \( f(x) = 2x - 1 \) and \( g(x) = x^2 + x - 2 \), find:

a. \((f - g)(x)\)

b. \((fg)(x)\)

c. \(\left(\frac{f}{g}\right)(x)\)

Determine the domain for each function.

Solution

a. \((f - g)(x) = f(x) - g(x)\)

\[= (2x - 1) - (x^2 + x - 2)\]

\[= 2x - 1 - x^2 - x + 2\]

\[= -x^2 + x + 1\]

This is the definition of the difference \( f - g \).

b. \((fg)(x) = f(x) \cdot g(x)\)

\[= (2x - 1)(x^2 + x - 2)\]

\[= 2x^3 + 2x^2 - 4x - x^2 - x + 2\]

\[= 2x^3 + x^2 - 5x + 2\]

This is the definition of the product \( fg \).

Because the equations for \( f \) and \( g \) do not involve division or contain even roots, the domain of both \( f \) and \( g \) is the set of all real numbers. Thus, the domain of \( f - g \) and \( fg \) is the set of all real numbers. However, for \( \frac{f}{g} \), the denominator cannot equal zero. We can factor the denominator as follows:

\[\left(\frac{f}{g}\right)(x) = \frac{2x - 1}{x^2 + x - 2}\]

\[= \frac{2x - 1}{(x + 2)(x - 1)}\]

We see that the domain for \( \frac{f}{g} \) is the set of all real numbers except \(-2\) and \(-1\): \(\{x | x \neq -2, x \neq 1\}\).

c. \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\)

\[= \frac{2x - 1}{x^2 + x - 2}\]

\[= \frac{2x - 1}{(x + 2)(x - 1)}\]

Because \( x + 2 \neq 0 \), \( x \neq -2 \)

Because \( x - 1 \neq 0 \), \( x \neq 1 \)

We see that the domain for \( \frac{f}{g} \) is the set of all real numbers except \(-2\) and \(-1\): \(\{x | x \neq -2, x \neq 1\}\).

Check Point 3

If \( f(x) = x - 5 \) and \( g(x) = x^2 - 1 \), find:

a. \((f - g)(x)\)

b. \((fg)(x)\)

c. \(\left(\frac{f}{g}\right)(x)\)

Determine the domain for each function.
Form composite functions.

**Composite Functions**

There is another way of combining two functions. To help understand this new combination, suppose that your computer store is having a sale. The models that are on sale cost either $300 less than the regular price or 85% of the regular price. If $x$ represents the computer’s regular price, both discounts can be described with the following functions:

\[ f(x) = x - 300 \quad \text{and} \quad g(x) = 0.85x. \]

At the store, you bargain with the salesperson. Eventually, she makes an offer you can’t refuse: The sale price is 85% of the regular price followed by a $300 reduction:

\[ 0.85x - 300. \]

In terms of functions $f$ and $g$, this offer can be obtained by taking the output of $g(x) = 0.85x$, namely $0.85x$, and using it as the input of $f$:

\[ f(x) = x - 300 \]

\[ f(0.85x) = 0.85x - 300. \]

Because $0.85x$ is $g(x)$, we can write this last equation as

\[ f(g(x)) = 0.85x - 300. \]

We read this equation as “$f$ of $g$ of $x$ is equal to $0.85x - 300.” We call $f(g(x))$ the composition of the function $f$ with $g$, or a composite function. This composite function is written $f \circ g$. Thus,

\[ (f \circ g)(x) = f(g(x)) = 0.85x - 300. \]

Like all functions, we can evaluate $f \circ g$ for a specified value of $x$ in the function’s domain. For example, here’s how to find the value of this function at 1400:

\[ (f \circ g)(x) = 0.85x - 300 \quad \text{This composite function describes the offer you cannot refuse.} \]

\[ (f \circ g)(1400) = 0.85(1400) - 300 = 1190 - 300 = 890. \]

This means that a computer that regularly sells for $1400 is on sale for $890 subject to both discounts.

Before you run out to buy a new computer, let’s generalize our discussion of the computer’s double discount and define the composition of any two functions.
EXAMPLE 4 Forming Composite Functions

Given and find:

a. 

b. 

Solution

a. We begin with \((f \circ g)(x)\), the composition of \(f\) with \(g\). Because \((f \circ g)(x)\) means \(f(g(x))\), we must replace each occurrence of \(x\) in the equation for \(f\) with \(g(x)\).

\[
(f \circ g)(x) = f(g(x)) = 3g(x) - 4
\]

This is the given equation for \(f\).

Replace \(x\) with \(g(x)\).

\[
(f \circ g)(x) = 3g(x) - 4
\]

Because \(g(x) = x^2 + 6\), replace \(g(x)\) with \(x^2 + 6\).

\[
= 3(x^2 + 6) - 4
\]

Use the distributive property.

\[
= 3x^2 + 18 - 4
\]

Simplify.

\[
= 3x^2 + 14
\]

Thus, \((f \circ g)(x) = 3x^2 + 14\).
Next, we find the composition of \( g \) with \( f \). Because \( (g \circ f)(x) \) means \( g(f(x)) \), we must replace each occurrence of \( x \) in the equation for \( g \) with \( f(x) \).

\[
g(x) = x^2 + 6 \hspace{1cm} \text{This is the given equation for } g.
\]

Replace \( x \) with \( f(x) \).

\[
(g \circ f)(x) = g(f(x)) = (f(x))^2 + 6
\]

Because \( f(x) = 3x - 4 \), replace \( f(x) \) with \( 3x - 4 \).

\[
= (3x - 4)^2 + 6
= 9x^2 - 24x + 16 + 6
= 9x^2 - 24x + 22
\]

Simplify.

Thus, \( (g \circ f)(x) = 9x^2 - 24x + 22 \). Notice that \( (g \circ g)(x) \) is not the same function as \( (g \circ f)(x) \).

Given \( f(x) = 5x + 6 \) and \( g(x) = x^2 - 1 \), find:

a. \( (f \circ g)(x) \)  

b. \( (g \circ f)(x) \).

We need to be careful in determining the domain for the composite function

\[
(f \circ g)(x) = f(g(x)).
\]

The following values must be excluded from the input \( x \):

- If \( x \) is not in the domain of \( g \), it must not be in the domain of \( f \circ g \).
- Any \( x \) for which \( g(x) \) is not in the domain of \( f \) must not be in the domain of \( f \circ g \).

**EXAMPLE 5**  **Forming a Composite Function and Finding Its Domain**

Given \( f(x) = \frac{2}{x - 1} \) and \( g(x) = \frac{3}{x} \), find:

a. \( (f \circ g)(x) \)  

b. the domain of \( f \circ g \).

**Solution**

a. Because \( (f \circ g)(x) \) means \( f(g(x)) \), we must replace \( x \) in \( f(x) = \frac{2}{x - 1} \) with \( g(x) \).

\[
(f \circ g)(x) = f(g(x)) = \frac{2}{g(x) - 1} = \frac{2}{\frac{3}{x} - 1} = \frac{2}{\frac{3}{x} - 1} \cdot \frac{x}{x} = \frac{2x}{3 - x}
\]

Thus, \( (f \circ g)(x) = \frac{2x}{3 - x} \).
b. We determine the domain of \((f \circ g)(x)\) in two steps.

<table>
<thead>
<tr>
<th>Rules for Excluding Numbers from the Domain of ((f \circ g)(x) = f(g(x)))</th>
<th>Applying the Rules to the Domain of and (f) (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>If (x) is not in the domain of (g), it must not be in the domain of (f \circ g).</td>
<td>The domain of (f) is ({x</td>
</tr>
<tr>
<td>Any (x) for which (g(x)) is not in the domain of (f) must not be in the domain of (f \circ g).</td>
<td>The domain of (g) is ({x</td>
</tr>
</tbody>
</table>

The domain of \(f \circ g\) is \(\{x| x \neq 0 \text{ and } x \neq 3\}\).

**Check Point 5**

Given \(f(x) = \frac{4}{x + 2}\) and \(g(x) = \frac{1}{x}\), find:

a. \((f \circ g)(x)\)

b. The domain of \(f \circ g\).

### Decomposing Functions

When you form a composite function, you “compose” two functions to form a new function. It is also possible to reverse this process. That is, you can “decompose” a given function and express it as a composition of two functions. Although there is more than one way to do this, there is often a “natural” selection that comes to mind first. For example, consider the function \(h\) defined by

\[
h(x) = (3x^2 - 4x + 1)^5.
\]

The function \(h\) takes \(3x^2 - 4x + 1\) and raises it to the power 5. A natural way to write \(h\) as a composition of two functions is to raise the function \(g(x) = 3x^2 - 4x + 1\) to the power 5. Thus, if we let

\[
f(x) = x^5 \text{ and } g(x) = 3x^2 - 4x + 1,
\]

then

\[
(f \circ g)(x) = f(g(x)) = f(3x^2 - 4x + 1) = (3x^2 - 4x + 1)^5.
\]

**Example 6**

**Writing a Function as a Composition**

Express as a composition of two functions:

\[
h(x) = \sqrt[5]{x^2 + 1}.
\]

**Solution** The function \(h\) takes \(x^2 + 1\) and takes its cube root. A natural way to write \(h\) as a composition of two functions is to take the cube root of the function \(g(x) = x^2 + 1\). Thus, we let

\[
f(x) = \sqrt[3]{x} \text{ and } g(x) = x^2 + 1.
\]
Exercise Set 2.6 • 257

We can check this composition by finding \((f \circ g)(x)\). This should give the original function, namely \(h(x) = \sqrt{x^2 + 1}\).

\[(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1} = h(x)\]

Express as a composition of two functions:

\(h(x) = \sqrt{x^2 + 5}\).

EXERCISE SET 2.6

Practice Exercises

1. If \(f(x) = 2x^2 - 5\) and \(g(x) = 3x + 7\), find:
   a. \((f + g)(x)\)
   b. \((f + g)(4)\).

2. If \(f(x) = 3x^2 - 2x + 1\) and \(g(x) = 4x - 1\), find:
   a. \((f + g)(x)\)
   b. \((f + g)(5)\).

3. Let \(f(x) = \sqrt{x - 6}\) and \(g(x) = \sqrt{x + 2}\), find:
   a. \((f + g)(x)\)
   b. the domain of \(f + g\).

4. Let \(f(x) = \sqrt{x - 8}\) and \(g(x) = \sqrt{x + 5}\), find:
   a. \((f + g)(x)\)
   b. the domain of \(f + g\).

In Exercises 5–16, find \(f + g\), \(f - g\), \(fg\), and \(\frac{f}{g}\). Determine the domain for each function.

5. \(f(x) = 2x + 3\), \(g(x) = x - 1\)
6. \(f(x) = 3x - 4\), \(g(x) = x + 2\)
7. \(f(x) = x - 5\), \(g(x) = 3x^2\)
8. \(f(x) = x - 6\), \(g(x) = 5x^2\)
9. \(f(x) = 2x^2 - x - 3\), \(g(x) = x + 1\)
10. \(f(x) = 6x^2 - x - 1\), \(g(x) = x - 1\)
11. \(f(x) = \sqrt{x}\), \(g(x) = x - 4\)
12. \(f(x) = \frac{x}{x}\), \(g(x) = x - 5\)
13. \(f(x) = 2 + \frac{1}{x}\), \(g(x) = \frac{1}{x}\)
14. \(f(x) = 6 - \frac{1}{x}\), \(g(x) = \frac{1}{x}\)
15. \(f(x) = \sqrt{x + 4}\), \(g(x) = \sqrt{x - 1}\)
16. \(f(x) = \sqrt{x + 6}\), \(g(x) = \sqrt{x - 3}\)

In Exercises 17–28, find:

a. \((f \circ g)(x)\)
   b. \((g \circ f)(x)\)
   c. \((f \circ g)(2)\).

17. \(f(x) = 2x\), \(g(x) = x + 7\)
18. \(f(x) = 3x\), \(g(x) = x - 5\)
19. \(f(x) = x + 4\), \(g(x) = 2x + 1\)
20. \(f(x) = 5x + 2\), \(g(x) = 3x - 4\)
21. \(f(x) = 4x - 3\), \(g(x) = 5x^2 - 2\)
22. \(f(x) = 7x + 1\), \(g(x) = 2x^2 - 9\)
23. \(f(x) = x^2 + 2\), \(g(x) = x^2 - 2\)
24. \(f(x) = x^2 + 1\), \(g(x) = x^2 - 3\)
25. \(f(x) = \sqrt{x}\), \(g(x) = x - 1\)
26. \(f(x) = \sqrt{x}\), \(g(x) = x + 2\)
27. \(f(x) = 2x - 3\), \(g(x) = \frac{x + 3}{2}\)
28. \(f(x) = 6x - 3\), \(g(x) = \frac{x + 3}{6}\)

In Exercises 29–38, find:

a. \((f \circ g)(x)\)
   b. the domain of \(f \circ g\).

29. \(f(x) = \frac{2}{x + 3}\), \(g(x) = \frac{1}{x}\)
30. \(f(x) = \frac{5}{x + 4}\), \(g(x) = \frac{1}{x}\)
31. \(f(x) = \frac{x}{x + 1}\), \(g(x) = \frac{4}{x}\)
32. \(f(x) = \frac{x}{x + 5}\), \(g(x) = \frac{6}{x}\)
33. \(f(x) = \sqrt{x}\), \(g(x) = x + 3\)
34. \(f(x) = \sqrt{x}\), \(g(x) = x - 3\)
35. \(f(x) = x^2 + 4\), \(g(x) = \sqrt{1 - x}\)
36. \(f(x) = x^2 + 1\), \(g(x) = \sqrt{2 - x}\)
37. \(f(x) = 4 - x^2\), \(g(x) = \sqrt{x^2 - 4}\)
38. \(f(x) = 9 - x^2\), \(g(x) = \sqrt{x^2 - 9}\)

In Exercises 39–46, express the given function \(h\) as a composition of two functions \(f\) and \(g\) so that \(h(x) = (f \circ g)(x)\).

39. \(h(x) = (3x - 1)^4\)
40. \(h(x) = (2x - 5)^3\)
41. \(h(x) = \sqrt{x^2 - 9}\)
42. \(h(x) = \sqrt{5x^2 + 3}\)
43. \(h(x) = |2x - 5|\)
44. \(h(x) = |3x - 4|\)
45. \(h(x) = \frac{1}{2x - 3}\)
46. \(h(x) = \frac{1}{4x + 5}\)
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In Exercises 47–58, use the graphs of \( f \) and \( g \) to evaluate each function.

\[ y = f(g(x)) \]

\[ y = g(f(x)) \]

47. \((f + g)(-3)\) 48. \((f + g)(-4)\)
49. \((f - g)(2)\) 50. \((g - f)(2)\)
51. \((f \div g)(-6)\) 52. \((f \div g)(-5)\)
53. \((fg)(-4)\) 54. \((fg)(-2)\)
55. \((f \circ g)(2)\) 56. \((f \circ g)(1)\)
57. \((g \circ f)(0)\) 58. \((g \circ f)(-1)\)

**Application Exercises**

It seems that Phideau’s medical bills are costing us an arm and a paw. The graph shows veterinary costs, in billions of dollars, for dogs and cats in five selected years. Let

\[ D(x) = \text{veterinary costs, in billions of dollars, for dogs in year } x \]
\[ C(x) = \text{veterinary costs, in billions of dollars, for cats in year } x. \]

**Use the graph to solve Exercises 59–62.**

<table>
<thead>
<tr>
<th>Year</th>
<th>Dogs</th>
<th>Cats</th>
</tr>
</thead>
<tbody>
<tr>
<td>1983</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1987</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>1991</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>1996</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>2000</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

*Source: American Veterinary Medical Association*

59. Find an estimate of \((D + C)(2000)\). What does this mean in terms of the variables in this situation?

60. Find an estimate of \((D - C)(2000)\). What does this mean in terms of the variables in this situation?

61. Using the information shown in the graph, what is the domain of \(D + C\)?

62. Using the information shown in the graph, what is the domain of \(D - C\)?

Consider the following functions:

\[ f(x) = \text{population of the world’s more developed regions in year } x \]
\[ g(x) = \text{population of the world’s less developed regions in year } x \]
\[ h(x) = \text{total world population in year } x. \]

Use these functions and the graph shown to answer Exercises 63–66.

63. What does the function \(f + g\) represent?
64. What does the function \(h - g\) represent?
65. Use the graph to estimate \((f + g)(2000)\).
66. Use the graph to estimate \((h - g)(2000)\).

67. A company that sells radios has a yearly fixed cost of $600,000. It costs the company $45 to produce each radio. Each radio will sell for $65. The company’s costs and revenue are modeled by the following functions:

\[ C(x) = 600,000 + 45x \quad \text{This function models the company’s costs.} \]
\[ R(x) = 65x. \quad \text{This function models the company’s revenue.} \]

Find and interpret \((R - C)(20,000)\), \((R - C)(30,000)\) and \((R - C)(40,000)\).
68. A department store has two locations in a city. From 1998 through 2002, the profits for each of the store’s two branches are modeled by the functions $f(x) = -0.44x + 13.62$ and $g(x) = 0.51x + 11.14$. In each model, $x$ represents the number of years after 1998 and $f$ and $g$ represent the profit, in millions of dollars.
   a. What is the slope of $f$? Describe what this means.
   b. What is the slope of $g$? Describe what this means.
   c. Find $f + g$. What is the slope of this function? What does this mean?

69. The regular price of a computer is $x$ dollars. Let $f(x) = x - 400$ and $g(x) = 0.75x$.
   a. Describe what the functions $f$ and $g$ model in terms of the price of the computer.
   b. Find $(f \circ g)(x)$ and describe what this models in terms of the price of the computer.
   c. Repeat part (b) for $(g \circ f)(x)$.
   d. Which composite function models the greater discount on the computer, $f \circ g$ or $g \circ f$? Explain.

70. The regular price of a pair of jeans is $x$ dollars. Let $f(x) = x - 5$ and $g(x) = 0.6x$.
   a. Describe what functions $f$ and $g$ model in terms of the price of the jeans.
   b. Find $(f \circ g)(x)$ and describe what this models in terms of the price of the jeans.
   c. Repeat part (b) for $(g \circ f)(x)$.
   d. Which composite function models the greater discount on the jeans, $f \circ g$ or $g \circ f$? Explain.

**Writing in Mathematics**

71. If equations for functions $f$ and $g$ are given, explain how to find $f + g$.

72. If the equations of two functions are given, explain how to obtain the quotient function and its domain.

73. If equations for functions $f$ and $g$ are given, describe two ways to find $(f - g)(3)$.

74. Explain how to use the graphs in Figure 2.53 on page 248 to estimate the per capita federal tax for any one of the years shown on the horizontal axis.

75. Describe a procedure for finding $(f \circ g)(x)$. What is the name of this function?

76. Describe the values of $x$ that must be excluded from the domain of $(f \circ g)(x)$.

**Technology Exercises**

77. The function $f(t) = -0.14t^2 + 0.51t + 31.6$ models the U.S. population ages 65 and older, $f(t)$, in millions, $t$ years after 1990. The function $g(t) = 0.54t^2 + 12.64t + 107.1$ models the total yearly cost of Medicare, $g(t)$, in billions of dollars, $t$ years after 1990. Graph the function $\frac{g}{f}$ in a $[0, 15, 1]$ by $[0, 60, 1]$ viewing rectangle. What does the shape of the graph indicate about the per capita costs of Medicare for the U.S. population ages 65 and over with increasing time?

78. Graph $y_1 = x^2 - 2x$, $y_2 = x$, and $y_3 = y_1 + y_2$ in the same $[-10, 10, 1]$ by $[-10, 10, 1]$ viewing rectangle. Then use the TRACE feature to trace along $y_3$. What happens at $x = 0$? Explain why this occurs.

79. Graph $y_1 = x^2 - 4$, $y_2 = \sqrt{4 - x^2}$, and $y_3 = y_2^2 - 4$ in the same $[-5, 5, 1]$ by $[-5, 5, 1]$ viewing rectangle. If $y_1$ represents $f$ and $y_2$ represents $g$, use the graph of $y_3$ to find the domain of $f \circ g$. Then verify your observation algebraically.

**Critical Thinking Exercises**

80. Which one of the following is true?
   a. If $f(x) = x^2 - 4$ and $g(x) = \sqrt{x^2 - 4}$, then $(f \circ g)(x) = -x^2$ and $(g \circ f)(5) = -25$.
   b. There can never be two functions $f$ and $g$, where $f \neq g$, for which $(f \circ g)(x) = (g \circ f)(x)$.
   c. If $f(7) = 5$ and $g(4) = 7$ then $(f \circ g)(4) = 35$.
   d. If $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$, then $(g \circ f)(5) = g(2)$.

81. Prove that if $f$ and $g$ are even functions, then $fg$ is also an even function.

82. Define two functions $f$ and $g$ so that $f \circ g = g \circ f$.

83. Use the graphs given in Exercises 63–66 to create a graph that shows the population, in billions, of less developed regions from 1950 through 2050.

**Group Exercise**

84. Consult an almanac, newspaper, magazine, or the Internet to find data displayed in a graph in the style of Figure 2.53 on page 248. Using the two graphs that group members find most interesting, introduce two functions that are related to the graphs. Then write and solve a problem involving function subtraction for each selected graph. If you are not sure where to begin, reread page 248–249 or look at Exercises 63–66 in this exercise set.
In most societies, women say they prefer to marry men who are older than themselves, whereas men say they prefer women who are younger. Evolutionary psychologists attribute these preferences to female concern with a partner’s material resources and male concern with a partner’s fertility (Source: David M. Buss, *Psychological Inquiry*, 6, 1–30). When the man is considerably older than the woman, people rarely comment. However, when the woman is older, as in the relationship between actors Susan Sarandon and Tim Robbins, people take notice.

Figure 2.56 shows the preferred age in a mate in five selected countries. We can focus on the data for the women and define a function.

![Preferred Age in a Mate](image)

**Figure 2.56**


Let the domain of the function be the set of the five countries shown in the graph. Let the range be the set of the average number of years women in each of the respective countries prefer men who are older than themselves. The function can be written as follows:

\[ f: \{(\text{Zambia}, 4.2), (\text{Colombia}, 4.5), (\text{Poland}, 3.3), (\text{Italy}, 3.3), (\text{U.S.}, 2.5)\} \]
Now let’s “undo” \( f \) by interchanging the first and second components in each of its ordered pairs. Switching the inputs and outputs of \( f \), we obtain the following relation:

\[
\text{Undoing } f: \{(4.2, \text{Zambia}), (4.5, \text{Colombia}), (3.3, \text{Poland}), (3.3, \text{Italy}), (2.5, \text{U.S.})\}.
\]

Can you see that this relation is not a function? Two of its ordered pairs have the same first component and different second components. This violates the definition of a function.

If a function \( f \) is a set of ordered pairs, \((x, y)\), then the changes produced by \( f \) can be “undone” by reversing the components of all the ordered pairs. The resulting relation, \((y, x)\), may or may not be a function. In this section, we will develop these ideas by studying functions whose compositions have a special “undoing” relationship.

**Inverse Functions**

Here are two functions that describe situations related to the price of a computer, \( x \):

\[
f(x) = x - 300 \quad g(x) = x + 300.
\]

Function \( f \) subtracts $300 from the computer’s price and function \( g \) adds $300 to the computer’s price. Let’s see what \( f(g(x)) \) does. Put \( g(x) \) into \( f \):

\[
f(g(x)) = g(x) - 300 \quad \text{This is the given equation for } f.
\]

Replace \( x \) with \( g(x) \).

\[
f(g(x)) = g(x) - 300 = x + 300 - 300 \quad \text{Because } g(x) = x + 300, \text{ replace } g(x) \text{ with } x + 300.
\]

\[
= x. \quad \text{This is the computer’s original price.}
\]

Using \( f(g(x)) \), the computer’s price, \( x \), went through two changes: the first, an increase; the second, a decrease:

\[
x + 300 - 300.
\]

The final price of the computer, \( x \), is identical to its starting price, \( x \).

In general, if the changes made to \( x \) by function \( g \) are undone by the changes made by function \( f \), then

\[
f(g(x)) = x.
\]

Assume, also, that this “undoing” takes place in the other direction:

\[
g(f(x)) = x.
\]

Under these conditions, we say that each function is the inverse function of the other. The fact that \( g \) is the inverse of \( f \) is expressed by renaming \( g \) as \( f^{-1} \), read “\( f \)-inverse.” For example, the inverse functions

\[
f(x) = x - 300 \quad g(x) = x + 300
\]

are usually named as follows:

\[
f(x) = x - 300 \quad f^{-1}(x) = x + 300.
\]
With these ideas in mind, we present the formal definition of the inverse of a function:

Definition of the Inverse of a Function

Let $f$ and $g$ be two functions such that

$$f(g(x)) = x \quad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for every } x \text{ in the domain of } f.$$ 

The function $g$ is the inverse of the function $f$, and is denoted by $f^{-1}$ (read "$f$-inverse"). Thus, the domain of $f$ is equal to the range of $f^{-1}$, and vice versa.

EXAMPLE 1 Verifying Inverse Functions

Show that each function is an inverse of the other:

$$f(x) = 5x \quad \text{and} \quad g(x) = \frac{x}{5}.$$ 

Solution To show that $f$ and $g$ are inverses of each other, we must show that $f(g(x)) = x$ and $g(f(x)) = x$. We begin with $f(g(x))$.

$$f(g(x)) = 5g(x) = 5\left(\frac{x}{5}\right) = x$$ 

Next, we find $g(f(x))$.

$$g(f(x)) = \frac{f(x)}{5} = \frac{5x}{5} = x$$

Because $g$ is the inverse of $f$ (and vice versa), we can use inverse notation and write

$$f(x) = 5x \quad \text{and} \quad f^{-1}(x) = \frac{x}{5}.$$ 

Notice how $f^{-1}$ undoes the change produced by $f$: $f$ changes $x$ by multiplying by 5 and $f^{-1}$ undoes this by dividing by 5.

Show that each function is an inverse of the other:

$$f(x) = 7x \quad \text{and} \quad g(x) = \frac{x}{7}.$$
EXAMPLE 2 Verifying Inverse Functions

Show that each function is an inverse of the other:

\[ f(x) = 3x + 2 \quad \text{and} \quad g(x) = \frac{x - 2}{3}. \]

Solution To show that \( f \) and \( g \) are inverses of each other, we must show that \( f(g(x)) = x \) and \( g(f(x)) = x \). We begin with \( f(g(x)) \).

\[
f(g(x)) = 3g(x) + 2 = 3 \left( \frac{x - 2}{3} \right) + 2 = x - 2 + 2 = x
\]

Next, we find \( g(f(x)) \).

\[
g(f(x)) = \frac{x - 2}{3} \quad \text{This is the equation for } g.
\]

\[
g(f(x)) = \frac{f(x) - 2}{3} = \frac{(3x + 2) - 2}{3} = \frac{3x}{3} = x
\]

Because \( g \) is the inverse of \( f \) (and vice versa), we can use inverse notation and write

\[ f(x) = 3x + 2 \quad \text{and} \quad f^{-1}(x) = \frac{x - 2}{3}. \]

Notice how \( f^{-1} \) undoes the changes produced by \( f \); \( f \) changes \( x \) by multiplying by 3 and adding 2, and \( f^{-1} \) undoes this by subtracting 2 and dividing by 3. This “undoing” process is illustrated in Figure 2.57.

Study Tip

Find the inverse of a function.

The procedure for finding a function’s inverse uses a switch-and-solve strategy. Switch \( x \) and \( y \), then solve for \( y \).
EXAMPLE 3 Finding the Inverse of a Function

Find the inverse of \( f(x) = 7x - 5 \).

Solution

Step 1 Replace \( f(x) \) with \( y \):

\[ y = 7x - 5 \]

Step 2 Interchange \( x \) and \( y \):

\[ x = 7y - 5 \]

This is the inverse function.

Step 3 Solve for \( y \):

\[ x + 5 = 7y \]

Add 5 to both sides.

\[ \frac{x + 5}{7} = y \]

Divide both sides by 7.

Step 4 Replace \( y \) with \( f^{-1}(x) \):

\[ f^{-1}(x) = \frac{x + 5}{7} \]

The equation is written with \( f^{-1} \) on the left.

Thus, the inverse of \( f(x) = 7x - 5 \) is \( f^{-1}(x) = \frac{x + 5}{7} \).

The inverse function, \( f^{-1} \), undoes the changes produced by \( f \). \( f \) changes \( x \) by multiplying by 7 and subtracting 5. \( f^{-1} \) undoes this by adding 5 and dividing by 7.

Check Point Find the inverse of \( f(x) = 2x + 7 \).

EXAMPLE 4 Finding the Equation of the Inverse

Find the inverse of \( f(x) = x^3 + 1 \).

Solution

Step 1 Replace \( f(x) \) with \( y \):

\[ y = x^3 + 1 \]

Step 2 Interchange \( x \) and \( y \):

\[ x = y^3 + 1 \]

Step 3 Solve for \( y \):

\[ x - 1 = y^3 \]

\[ \sqrt[3]{x - 1} = \sqrt[3]{y^3} \]

\[ \sqrt[3]{x - 1} = y \]

Step 4 Replace \( y \) with \( f^{-1}(x) \):

\[ f^{-1}(x) = \sqrt[3]{x - 1} \]

Thus, the inverse of \( f(x) = x^3 + 1 \) is \( f^{-1}(x) = \sqrt[3]{x - 1} \).

Check Point Find the inverse of \( f(x) = 4x^3 - 1 \).
Use the horizontal line test to determine if a function has an inverse function.

**The Horizontal Line Test and One-to-One Functions**

Let's see what happens if we try to find the inverse of the standard quadratic function, \( f(x) = x^2 \).

**Step 1** Replace \( f(x) \) with \( y \): \( y = x^2 \).

**Step 2** Interchange \( x \) and \( y \): \( x = y^2 \).

**Step 3** Solve for \( y \): We apply the square root method to solve for \( y \).

We obtain \( y = \pm \sqrt{x} \).

The \( \pm \) in this last equation shows that for certain values of \( x \) (all positive real numbers), there are two values of \( y \). Because this equation does not represent \( y \) as a function of \( x \), the standard quadratic function does not have an inverse function.

Can we look at the graph of a function and tell if it represents a function with an inverse? Yes. The graph of the standard quadratic function is shown in Figure 2.58. Four units above the \( x \)-axis, a horizontal line is drawn. This line intersects the graph at two of its points, \((-2, 4)\) and \((2, 4)\). Because inverse functions have ordered pairs with the coordinates reversed, let’s see what happens if we reverse these coordinates. We obtain \((4, -2)\) and \((4, 2)\). A function provides exactly one output for each input. However, the input 4 is associated with two outputs, -2 and 2. The points \((4, -2)\) and \((4, 2)\) do not define a function.

If any horizontal line, such as the one in Figure 2.58, intersects a graph at two or more points, these points will not define a function when their coordinates are reversed. This suggests the **horizontal line test** for inverse functions:

**The Horizontal Line Test For Inverse Functions**

A function \( f \) has an inverse that is a function, \( f^{-1} \), if there is no horizontal line that intersects the graph of the function \( f \) at more than one point.

**EXAMPLE 5** Applying the Horizontal Line Test

Which of the following graphs represent functions that have inverse functions?

![Graphs](image_url)

**Solution** Can you see that horizontal lines can be drawn in parts (b) and (c) that intersect the graphs more than once? This is illustrated in the figure at the top of the next page. These graphs do not pass the horizontal line test. The graphs in parts (b) and (c) are not the graphs of functions with inverse functions. By contrast, no horizontal line can be drawn in parts (a) and (d) that intersect the graphs more than once. These graphs pass the horizontal line test. Thus, the graphs in parts (a) and (d) represent functions that have inverse functions.
Which of the following graphs represent functions that have inverse functions?

A function passes the horizontal line test when no two different ordered pairs have the same second component. This means that if \( x_1 \neq x_2 \), then \( f(x_1) \neq f(x_2) \). Such a function is called a one-to-one function. Thus, a one-to-one function is a function in which no two different ordered pairs have the same second component. Only one-to-one functions have inverse functions. Any function that passes the horizontal line test is a one-to-one function. Any one-to-one function has a graph that passes the horizontal line test.

**Graphs of \( f \) and \( f^{-1} \)**

There is a relationship between the graph of a one-to-one function, \( f \), and its inverse, \( f^{-1} \). Because inverse functions have ordered pairs with the coordinates reversed, if the point \((a, b)\) is on the graph of \( f \), then the point \((b, a)\) is on the graph of \( f^{-1} \). The points \((a, b)\) and \((b, a)\) are symmetric with respect to the line \( y = x \). Thus, the graph of \( f^{-1} \) is a reflection of the graph of \( f \) about the line \( y = x \). This is illustrated in Figure 2.59.

**Figure 2.59** The graph of \( f^{-1} \) is a reflection of the graph of \( f \) about the line \( y = x \).
EXAMPLE 6  Graphing the Inverse Function

Use the graph of \( f \) in Figure 2.60 to draw the graph of its inverse function.

**Solution**  We begin by noting that no horizontal line intersects the graph of \( f \) at more than one point, so \( f \) does have an inverse function. Because the points \((-3, -2), (-1, 0), \) and \((4, 2)\) are on the graph of \( f \), the graph of the inverse function, \( f^{-1} \), has points with these ordered pairs reversed. Thus, \((-2, -3), (0, -1)\), and \((2, 4)\) are on the graph of \( f^{-1} \). We can use these points to graph \( f^{-1} \). The graph of \( f^{-1} \) is shown in Figure 2.61. Note that the graph of \( f^{-1} \) is the reflection of the graph of \( f \) about the line \( y = x \).

**EXERCISE SET 2.7**

The functions in Exercises 11–30 are all one-to-one. For each function:

a. Find an equation for \( f^{-1}(x) \), the inverse function.

b. Verify that your equation is correct by showing that \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).

1. \( f(x) = 4x \) and \( g(x) = \frac{x}{4} \)
2. \( f(x) = 6x \) and \( g(x) = \frac{x}{6} \)
3. \( f(x) = 3x + 8 \) and \( g(x) = \frac{x - 8}{3} \)
4. \( f(x) = 4x + 9 \) and \( g(x) = \frac{x - 9}{4} \)
5. \( f(x) = 5x - 9 \) and \( g(x) = \frac{x + 9}{5} \)
6. \( f(x) = 3x - 7 \) and \( g(x) = \frac{x + 7}{3} \)
7. \( f(x) = \frac{3}{x - 4} \) and \( g(x) = \frac{3}{x} + 4 \)
8. \( f(x) = \frac{2}{x - 5} \) and \( g(x) = \frac{2}{x} + 5 \)
9. \( f(x) = -x \) and \( g(x) = -x \)
10. \( f(x) = \sqrt[3]{x - 4} \) and \( g(x) = x^3 + 4 \)
11. \( f(x) = x + 3 \) 12. \( f(x) = x + 5 \)
13. \( f(x) = 2x \) 14. \( f(x) = 4x \)
15. \( f(x) = 2x + 3 \) 16. \( f(x) = 3x - 1 \)
17. \( f(x) = x^3 + 2 \) 18. \( f(x) = x^3 - 1 \)
19. \( f(x) = (x + 2)^3 \) 20. \( f(x) = (x - 1)^3 \)
21. \( f(x) = \frac{1}{x} \) 22. \( f(x) = \frac{2}{x} \)
23. \( f(x) = \sqrt{x} \) 24. \( f(x) = \sqrt[3]{x} \)
25. \( f(x) = x^2 + 1 \), for \( x \geq 0 \) 26. \( f(x) = x^2 - 1 \), for \( x \geq 0 \)
27. \( f(x) = \frac{2x + 1}{x - 3} \) 28. \( f(x) = \frac{2x - 3}{x + 1} \)
29. \( f(x) = \sqrt[3]{x - 4} + 3 \) 30. \( f(x) = x^{3/3} \)
Which graphs in Exercises 31–36 represent functions that have inverse functions?

31. 
32. 
33. 
34. 
35. 
36. 

In Exercises 37–40, use the graph of \( f \) to draw the graph of its inverse function.

37. 
38. 

Application Exercises

41. Refer to Figure 2.56 on page 260. Recall that the bar graphs in the figure show the preferred age in a mate in five selected countries.

   a. Consider a function, \( f \), whose domain is the set of the five countries shown in the graph. Let the range be the set of the average number of years men in each of the respective countries prefer women who are younger than themselves. (You will need to use the graph to estimate these values. Assume that the bars for Poland and Italy have the same length. Round to the nearest tenth of a year.) Write function \( f \) as a set of ordered pairs.

   b. Write the relation that is the inverse of \( f \) as a set of ordered pairs. Is this relation a function? Explain your answer.

42. The bar graph shows the percentage of land owned by the federal government in western states in which the government owns at least half of the land.

   Percentage of Land Owned by the Federal Government

<table>
<thead>
<tr>
<th>State</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nevada</td>
<td>83%</td>
</tr>
<tr>
<td>Utah</td>
<td>65%</td>
</tr>
<tr>
<td>Idaho</td>
<td>62%</td>
</tr>
<tr>
<td>Alaska</td>
<td>62%</td>
</tr>
<tr>
<td>Oregon</td>
<td>52%</td>
</tr>
<tr>
<td>Wyoming</td>
<td>50%</td>
</tr>
</tbody>
</table>

   Source: Bureau of Land Management
**43.** The graph represents the probability of two people in the same room sharing a birthday as a function of the number of people in the room. Call the function \( f \).

- **a.** Explain why \( f \) has an inverse that is a function.
- **b.** Describe in practical terms the meaning of \( f^{-1}(0.25) \), \( f^{-1}(0.5) \), and \( f^{-1}(0.7) \).

**44.** The graph shows the average age at which women in the United States marry for the first time over a 110-year period.

- **a.** Does this graph have an inverse that is a function? What does this mean about the average age at which U.S. women marry during the period shown?
- **b.** Identify two or more years in which U.S. women married for the first time at the same average age. What is a reasonable estimate of this average age?

**Writing in Mathematics**

**47.** Explain how to determine if two functions are inverses of each other.

**48.** Describe how to find the inverse of a one-to-one function.

**49.** What is the horizontal line test and what does it indicate?

**50.** Describe how to use the graph of a one-to-one function to draw the graph of its inverse function.

**51.** How can a graphing utility be used to visually determine if two functions are inverses of each other?
2.2 Distance and Midpoint Formulas; Circles

DEFINITIONS AND CONCEPTS

a. The distance, \( d \), between the points \((x_1, y_1)\) and \((x_2, y_2)\) is given by
\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

b. The midpoint of the line segment whose endpoints are \((x_1, y_1)\) and \((x_2, y_2)\) is the point with coordinates \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).

EXAMPLES

65. If \( f(x) = 3x \) and \( g(x) = x + 5 \), find \((f \circ g)^{-1}(x)\) and \((g^{-1} \circ f^{-1})(x)\).

66. Show that \( f(x) = \frac{3x - 2}{5x - 3} \) is its own inverse.

67. Freedom 7 was the spacecraft that carried the first American into space in 1961. Total flight time was 15 minutes, and the spacecraft reached a maximum height of 116 miles. Consider a function, \( s(t) \), that expresses Freedom 7’s height, \( s(t) \), in miles, after \( t \) minutes. Is \( s \) a one-to-one function? Explain your answer.

68. If \( f(2) = 6 \), find \( x \) satisfying \( 8 + f^{-1}(x - 1) = 10 \).

CRITICAL THINKING EXERCISES

64. Which one of the following is true?
   a. The inverse of \( \{(1, 4), (2, 7)\} \) is \( \{(2, 7), (1, 4)\} \).
   b. The function \( f(x) = 5 \) is one-to-one.
   c. If \( f(x) = 3x \), then \( f^{-1}(x) = \frac{1}{3x} \).
   d. The domain of \( f \) is the same as the range of \( f^{-1} \).

CHAPTER SUMMARY, REVIEW, AND TEST

Summary

DEFINITIONS AND CONCEPTS

2.1 Lines and Slope

a. The slope, \( m \), of the line through \((x_1, y_1)\) and \((x_2, y_2)\) is
\[
m = \frac{y_2 - y_1}{x_2 - x_1}.
\]

b. Equations of lines include point-slope form, \( y - y_1 = m(x - x_1) \), slope-intercept form, \( y = mx + b \), and general form, \( Ax + By + C = 0 \). The equation of a horizontal line is \( y = b \); a vertical line is \( x = a \).

c. Parallel lines have equal slopes. Perpendicular lines have slopes that are negative reciprocals.

Ex. 1, p.177

Ex. 2 & 3, p.179;

Ex. 5 & 6, p.182

Ex. 8 & 9, p.184–185

2.2 Distance and Midpoint Formulas; Circles

a. The distance, \( d \), between the points \((x_1, y_1)\) and \((x_2, y_2)\) is given by
\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

b. The midpoint of the line segment whose endpoints are \((x_1, y_1)\) and \((x_2, y_2)\) is the point with coordinates \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).

Ex. 1, p.194

Ex. 2, p.195
c. The standard form of the equation of a circle with center \((h, k)\) and radius \(r\) is 
\[(x - h)^2 + (y - k)^2 = r^2.\]

b. The general form of the equation of a circle is 
\[x^2 + y^2 + Dx + Ey + F = 0.\]

d. To convert from the general form to the standard form of a circle’s equation, complete the square on \(x\) and \(y\).

e. The difference quotient is 
\[\frac{f(x + h) - f(x)}{h}, \quad h \neq 0.\]

e. If a function \(f\) does not model data or verbal conditions, its domain is the largest set of real numbers for which the value of \(f(x)\) is a real number. Exclude from the function’s domain real numbers that cause division by zero and real numbers that result in an even root of a negative number.

2.3 Basics of Functions

a. A relation is any set of ordered pairs. The set of first components is the domain and the set of second components is the range.

b. A function is a correspondence from a first set, called the domain, to a second set, called the range, such that each element in the domain corresponds to exactly one element in the range. If any element in a relation’s domain corresponds to more than one element in the range, the relation is not a function.

c. Functions are usually given in terms of equations involving \(x\) and \(y\), in which \(x\) is the independent variable and \(y\) is the dependent variable. If an equation is solved for \(y\) and more than one value of \(y\) can be obtained for a given \(x\), then the equation does not define \(y\) as a function of \(x\). If an equation defines a function the value of the function at \(x\), \(f(x)\), often replaces \(y\).

d. The difference quotient is 
\[\frac{f(x + h) - f(x)}{h}, \quad h \neq 0.\]

e. If the graph of a function is given, we can often visually locate the number(s) at which the function is undefined or discontinuous. Figures 2.32, p. 222.

2.4 Graphs of Functions

a. The graph of a function is the graph of its ordered pairs.

b. The vertical line test for functions: If any vertical line intersects a graph in more than one point, the graph does not define \(y\) as a function of \(x\).

c. A function is increasing on intervals where its graph rises, decreasing on intervals where it falls, and constant on intervals where it neither rises nor falls. Precise definitions are given in the box on page 220.

d. If the graph of a function is given, we can often visually locate the number(s) at which the function has a relative maximum or relative minimum. Figures 2.21, p. 222;

2.5 Transformations of Functions

a. Table 2.4 on pages 235–236 shows the graphs of the constant function, \(f(x) = c\), the identity function, \(f(x) = x\), the standard quadratic function, \(f(x) = x^2\), the standard cubic function, \(f(x) = x^3\), the square root function, \(f(x) = \sqrt{x}\), and the absolute value function, \(f(x) = |x|\). The table also lists characteristics of each function.

b. Table 2.5 on page 243 summarizes how to graph a function using vertical shifts, \(y = f(x) \pm c\), horizontal shifts, \(y = f(x \pm c)\), reflections about the \(x\)-axis, \(y = -f(x)\), reflections about the \(y\)-axis, \(y = f(-x)\), vertical stretching, \(y = cf(x), c > 1\), and vertical shrinking, \(y = cf(x), 0 < c < 1\).

c. A function involving more than one transformation can be graphed in the following order: (1) horizontal shifting; (2) vertical stretching or shrinking; (3) reflecting; (4) vertical shifting.
2.6 Combinations of Functions; Composite and Inverse Functions

a. When functions are given as equations, they can be added, subtracted, multiplied, or divided by performing operations with the algebraic expressions that appear on the right side of the equations. Ex. 1, p. 250; definitions for the sum \( f + g \), the difference \( f - g \), the product \( fg \), and the quotient \( \frac{f}{g} \) functions are given in the box on page 251.

b. The composition of functions \( f \) and \( g \), is defined by \( (f \circ g)(x) = f(g(x)) \). The domain of the composite function \( f \circ g \) is given in the box on page 256. This composite function is obtained by replacing each occurrence of \( x \) in the equation for \( f \) with \( g(x) \).

2.7 Inverse Functions

a. If and function \( g \) is the inverse of function \( f \), denoted \( f^{-1} \) and read “\( f \) inverse.” Thus, to show that \( f \) and \( g \) are inverses of each other, one must show \( f(g(x)) = x \) and \( g(f(x)) = x \).

b. The procedure for finding a function’s inverse uses a switch-and-solve strategy. Switch \( x \) and \( y \), then solve for \( y \). The procedure is given in the box on page 263.

c. The horizontal line test for inverse functions: A function \( f \) has an inverse that is a function, if there is no horizontal line that intersects the graph of the function \( f \) at more than one point.

d. A one-to-one function is one in which no two different ordered pairs have the same second component. Only one-to-one functions have inverse functions.

e. If the point \((a, b)\) is on the graph of \( f \), then the point \((b, a)\) is on the graph of \( \text{ref}(f) \) about the line \( y = x \).

\[
\begin{align*}
g(f(x)) &= x, \\
f(g(x)) &= x, \\
(f \circ g)(x) &= f(g(x)).
\end{align*}
\]

Review Exercises

2.1

In Exercises 1–4, find the slope of the line passing through each pair of points or state that the slope is undefined. Then indicate whether the line through the points rises, falls, is horizontal, or is vertical.

1. \((3, 2)\) and \((5, 1)\)
2. \((-1, -2)\) and \((-3, -4)\)
3. \((-3, \frac{3}{2})\) and \((6, \frac{3}{2})\)
4. \((-2, 5)\) and \((-2, 10)\)

In Exercises 5–6, use the given conditions to write an equation for each line in point-slope form and slope-intercept form.

5. Passing through \((-3, 2)\) with slope \(-6\)
6. Passing through \((1, 6)\) and \((-1, 2)\)

In Exercises 7–10, give the slope and y-intercept of each line whose equation is given. Then graph the line.

7. \(y = \frac{1}{2}x - 1\)
8. \(y = -4x + 5\)
9. \(2x + 3y + 6 = 0\)
10. \(2y - 8 = 0\)

Corporations in the United States are doing quite well, thank you. The scatter plot in the next column shows corporate profits, in billions of dollars, from 1990 through 2000. Also shown is a line that passes through or near the points.

Source: US Department of Labor

a. Use the two points whose coordinates are shown by the voice balloons to find the point-slope equation of the line that models corporate profits, \(y\), in billions of dollars, \(x\) years after 1990.

b. Write the equation in part (a) in slope-intercept form.

c. Use the linear model to predict corporate profits in 2010.

12. The scatter plot on the next page shows the number of minutes each that 16 people exercise per week and the number of headaches per month each person experiences.
In Exercises 24–26, determine whether each relation is a function. Give the domain and range for each relation.

24. \{(2, 7), (3, 7), (5, 7)\}  
25. \{(1, 10) (2, 500), (13, \pi)\}  
26. \{(12, 13), (14, 15), (12, 19)\}

In Exercises 27–29, determine whether each equation defines \(y\) as a function of \(x\).

27. \(2x + y = 8\)  
28. \(3x^2 + y = 14\)  
29. \(2x + y^2 = 6\)

In Exercises 30–33, evaluate each function at the given values of the independent variable and simplify.

30. \(f(x) = 5 - 7x\)  
   a. \(f(4)\)  
   b. \(f(x + 3)\)  
   c. \(f(-x)\)

31. \(g(x) = 3x^2 - 5x + 2\)  
   a. \(g(0)\)  
   b. \(g(-2)\)  
   c. \(g(x - 1)\)  
   d. \(g(-x)\)

32. \(g(x) = \begin{cases} \sqrt{x - 4} & \text{if } x \geq 4 \\ 4 - x & \text{if } x < 4 \end{cases}\)  
   a. \(g(13)\)  
   b. \(g(0)\)  
   c. \(g(-3)\)

33. \(f(x) = \begin{cases} x^2 - 1 & \text{if } x \neq 1 \\ 12 & \text{if } x = 1 \end{cases}\)  
   a. \(f(-2)\)  
   b. \(f(1)\)  
   c. \(f(2)\)

In Exercises 34–35, find and simplify the difference quotient \(\frac{f(x + h) - f(x)}{h}, \quad h \neq 0\) for the given function.

34. \(f(x) = 8x - 11\)  
35. \(f(x) = x^2 - 13x + 5\)

In Exercises 36–40, find the domain of each function.

36. \(f(x) = x^2 + 6x - 3\)  
37. \(g(x) = \frac{4}{x - 7}\)  
38. \(h(x) = \sqrt{8 - 2x}\)  
39. \(f(x) = \frac{x}{x^2 - 1}\)  
40. \(g(x) = \frac{\sqrt{x - 7}}{x - 5}\)

2.4

Graph the functions in Exercises 41–42. Use the integer values of \(x\) given to the right of the function to obtain the ordered pairs. Use the graph to specify the function’s domain and range.

41. \(f(x) = x^2 - 4x + 4\)  
   \(x = -1, 0, 1, 2, 3, 4\)

42. \(f(x) = |2 - x|\)  
   \(x = -1, 0, 1, 2, 3, 4\)

In Exercises 43–45, use the graph to determine a. the function’s domain; b. the function’s range; c. the \(x\)-intercepts, if any; d. the \(y\)-intercept, if any; e. intervals on which the function is increasing, decreasing, or constant; and f. the function values indicated below the graphs.

43. 

\[ f(-2) = \,? \quad f(3) = \,? \]
In Exercises 46–47, find:

a. The numbers, if any, at which \( f \) has a relative maximum. What are these relative maxima?
b. The numbers, if any, at which \( f \) has a relative minimum. What are these relative minima?

46. Use the graph in Exercise 43.
47. Use the graph in Exercise 44.

In Exercises 48–51, use the vertical line test to identify graphs in which \( y \) is a function of \( x \).

48. 

49. 

50. 

51. 

52. Find the average rate of change of \( f(x) = x^2 - 4x \) from \( x_1 = 5 \) to \( x_2 = 9 \).

53. The graph shows annual spending per uniformed member of the U.S. military in inflation-adjusted dollars. Find the average rate of change of spending per year from 1955 through 2000. Round to the nearest dollar per year.

In Exercises 54–56, determine whether each function is even, odd, or neither. State each function’s symmetry. If you are using a graphing utility, graph the function and verify its possible symmetry.

54. \( f(x) = x^3 - 5x \)
55. \( f(x) = x^4 - 2x^3 + 1 \)
56. \( f(x) = 2x\sqrt{1 - x^2} \)

57. The graph shows the height, in meters, of a vulture in terms of its time, in seconds, in flight.

a. Is the vulture’s height a function of time? Use the graph to explain why or why not.
b. On which interval is the function decreasing? Describe what this means in practical terms.
c. On which intervals is the function constant? What does this mean for each of these intervals?
d. On which interval is the function increasing? What does this mean?
58. A cargo service charges a flat fee of $5 plus $1.50 for each pound or fraction of a pound. Graph shipping cost, \( C(x) \), in dollars, as a function of weight, \( x \), in pounds, for \( 0 < x \leq 5 \).

2.5

In Exercises 59–61, begin by graphing the standard quadratic function, \( f(x) = x^2 \). Then use transformations of this graph to graph the given function.

59. \( g(x) = x^2 + 2 \)  \( h(x) = (x + 2)^2 \)
60. \( r(x) = -(x + 1)^2 \)

In Exercises 62–64, begin by graphing the square root function, \( f(x) = \sqrt{x} \). Then use transformations of this graph to graph the given function.

62. \( g(x) = \sqrt{x + 3} \)  \( h(x) = \sqrt{3 - x} \)
63. \( r(x) = 2\sqrt{x + 2} \)

In Exercises 65–67, begin by graphing the absolute value function, \( f(x) = |x| \). Then use transformations of this graph to graph the given function.

65. \( g(x) = |x + 2| - 3 \)  \( h(x) = -|x - 1| + 1 \)
66. \( r(x) = \frac{1}{3}|x + 2| \)

In Exercises 68–70, begin by graphing the standard cubic function, \( f(x) = x^3 \). Then use transformations of this graph to graph the given function.

68. \( g(x) = \frac{1}{2}(x - 1)^3 \)  \( h(x) = -(x + 1)^3 \)
70. \( r(x) = \frac{1}{4}x^3 - 1 \)

In Exercises 71–73, use the graph of the function \( f \) to sketch the graph of the given function \( g \).

71. \( g(x) = f(x + 2) + 3 \)  \( g(x) = \frac{1}{2}f(x - 1) \)
72. \( g(x) = -2 + 2f(x + 2) \)

2.6

In Exercises 74–76, find \( f + g \), \( f - g \), \( fg \), and \( \frac{f}{g} \). Determine the domain for each function.

74. \( f(x) = 3x - 1 \), \( g(x) = x - 5 \)
75. \( f(x) = x^2 + x + 1 \), \( g(x) = x^2 - 1 \)
76. \( f(x) = \sqrt{x} + 7 \), \( g(x) = \sqrt{x} - 2 \)

In Exercises 77–78, find a. \( (f \circ g)(x) \); b. \( (g \circ f)(x) \); c. \( (f \circ g)(3) \).

77. \( f(x) = x^2 + 3 \), \( g(x) = 4x - 1 \)
78. \( f(x) = \sqrt{x} \), \( g(x) = x + 1 \)

In Exercises 79–80, find a. \( (f \circ g)(x) \); b. the domain of \( (f \circ g) \).

79. \( f(x) = \frac{x + 1}{x - 2} \), \( g(x) = \frac{1}{x} \)
80. \( f(x) = \sqrt{x - 1} \), \( g(x) = x + 3 \)

In Exercises 81–82, express the given function \( h \) as a composition of two functions \( f \) and \( g \) so that \( h(x) = (f \circ g)(x) \).

81. \( h(x) = (x^2 + 2x - 1)^4 \)
82. \( h(x) = \sqrt{7x + 4} \)

2.7

In Exercises 83–84, find \( f(g(x)) \) and \( g(f(x)) \) and determine whether each pair of functions \( f \) and \( g \) are inverses of each other.

83. \( f(x) = \frac{3}{5}x + \frac{1}{2} \) and \( g(x) = \frac{5}{3}x - 2 \)
84. \( f(x) = 2 - 5x \) and \( g(x) = \frac{2 - x}{5} \)

The functions in Exercises 85–87 are all one-to-one. For each function:

a. Find an equation for \( f^{-1}(x) \), the inverse function.

b. Verify that your equation is correct by showing that \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).

85. \( f(x) = 4x - 3 \)
86. \( f(x) = \sqrt{x + 2} \)
87. \( f(x) = 8x^3 + 1 \)

Which graphs in Exercises 88–91 represent functions that have inverse functions?

88. [Graph]

89. [Graph]
Chapter 2 Test

In Exercises 1–2, use the given conditions to write an equation for each line in point-slope form and slope-intercept form.

1. Passing through (2, 1) and (−1, −8)
2. Passing through (−4, 6) and perpendicular to the line whose equation is \( y = -\frac{1}{4}x + 5 \)

3. Strong demand plus higher fuel and labor costs are driving up the price of flying. The graph shows the national averages for one-way fares. Also shown is a line that models that data.

4. Give the center and radius of the circle whose equation is \( x^2 + y^2 + 4x - 6y - 3 = 0 \) and graph the equation.

5. List by letter all relations that are not functions.
   a. \( \{ (7, 5), (8, 5), (9, 5) \} \)
   b. \( \{ (5, 7), (5, 8), (5, 9) \} \)
   c. \( y^2 = x^2 + 9 \)
   d. \( x^2 + y^2 = 100 \)
   e. \( y = x^2 + 2 \)

6. If \( f(x) = x^2 - 2x + 5 \), find \( f(x - 1) \) and simplify.
7. If \( g(x) = \begin{cases} \sqrt{x - 3} & \text{if } x \geq 3 \\ 3 - x & \text{if } x < 3 \end{cases} \), find \( g(-1) \) and \( g(7) \).
8. If \( f(x) = \sqrt{12 - 3x} \), find the domain of \( f \).
9. If \( f(x) = x^2 + 11x - 7 \), find and simplify the difference quotient \( \frac{f(x + h) - f(x)}{h} \).
10. Use the graph of function \( f \) to answer the following questions.

   - a. What is \( f(4) - f(-3) \)?
   - b. What is the domain of \( f \)?
   - c. What is the range of \( f \)?
   - d. On which interval or intervals is \( f \) increasing?
   - e. On which interval or intervals is \( f \) decreasing?
   - f. For what number does \( f \) have a relative maximum?
   - g. What is the relative maximum?
   - h. For what number does \( f \) have a relative minimum?
   - i. What is the relative minimum?

11. Find the average rate of change of \( f(x) = 3x^2 - 5 \) from \( x_1 = 6 \) to \( x_2 = 10 \).
12. Determine whether \( f(x) = x^4 - x^2 \) is even, odd, or neither. Use your answer to explain why the graph in the figure shown cannot be the graph of \( f \).

13. The figure at the top of the next column shows how the graph of \( h(x) = -2(x - 3)^2 \) is obtained from the graph of \( f(x) = x^2 \). Describe this process, using the graph of \( g \) in your description.

14. Begin by graphing the absolute value function, \( f(x) = |x| \).
Then use transformations of this graph to graph \( g(x) = \frac{1}{2}|x + 1| + 3 \).
If \( f(x) = x^2 + 3x - 4 \) and \( g(x) = 5x - 2 \), find each function or function value in Exercises 15–19.
15. \( (f - g)(x) \)
16. \( \left( \frac{f - g}{h} \right)(x) \) and its domain
17. \( (f \circ g)(x) \)
18. \( (g \circ f)(x) \)
19. \( f(g(2)) \)
20. If \( f(x) = \frac{7}{x - 4} \) and \( g(x) = \frac{2}{x} \), find \( (f \circ g)(x) \) and the domain of \( f \circ g \).
21. Express \( h(x) = (2x + 13)^7 \) as a composition of two functions \( f \) and \( g \) so that \( h(x) = (f \circ g)(x) \).
22. If \( f(x) = \sqrt{x - 2} \), find the equation for \( f^{-1}(x) \). Then verify that your equation is correct by showing that \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).
23. A function \( f \) models the amount given to charity as a function of income. The graph of \( f \) is shown in the figure.

   - a. Explain why \( f \) has an inverse that is a function.
   - b. Find \( f(80) \).
   - c. Describe in practical terms the meaning of \( f^{-1}(2000) \).
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24. Use a graphing utility to graph $f(x) = \frac{x^3}{3} + x^2 - 15x + 3$ in a $[-10, 10, 1]$ by $[-30, 70, 10]$ viewing rectangle. Use the graph to answer the following questions.
   a. Is $f$ one-to-one? Explain.
   b. Is $f$ even, odd, or neither? Explain.
   c. What is the range of $f$?

Cumulative Review Exercises (Chapters P–2)

Simplify each expression in Exercises 1 and 2.

1. $\frac{4x^3y}{2x^3y^3}$
2. $\frac{5}{4\sqrt{2}}$

3. Factor: $x^3 - 4x^2 + 2x - 8$.

In Exercises 4 and 5, perform the operations and simplify.

4. $\frac{x - 3}{x + 4} + \frac{x}{x - 2}$
5. $\frac{4 + 2}{x}$

Solve each equation in Exercises 6–9.

6. $(x + 3)(x - 4) = 8$
7. $3(4x - 1) = 4 - 6(x - 3)$
8. $\sqrt{x} + 2 = x$
9. $x^{2/3} - x^{1/3} - 6 = 0$

Solve each inequality in Exercises 10 and 11. Express the answer in interval notation.

10. $\frac{x}{2} - 3 \leq \frac{x}{4} + 2$
11. $\frac{x + 3}{x - 2} \leq 2$

12. Write the point-slope form and the slope-intercept form of the line passing through $(-2, 5)$ and perpendicular to the line whose equation is $y = -\frac{1}{4}x + \frac{1}{2}$.

13. Graph $f(x) = \sqrt{x}$ and then use transformations of this graph to graph $g(x) = \sqrt{x - 3} + 4$ in the same rectangular coordinate system.

14. If $f(x) = 2 + \sqrt{x - 3}$, find the equation for $f^{-1}(x)$.

15. If $f(x) = 3 - x^2$, find $\frac{f(x + h) - f(x)}{h}$ and simplify.

16. Solve for $c$: $A = \frac{cd}{c + d}$.

17. You invested $6000 in two accounts paying 7% and 9% annual interest, respectively. At the end of the year, the total interest from these investments was $510. How much was invested at each rate?

18. For a summer sales job, you are choosing between two pay arrangements: a weekly salary of $200 plus 5% commission on sales, or a straight 15% commission. For how many dollars of sales will the earnings be the same regardless of the pay arrangement?

19. The length of a rectangular garden is 2 feet more than twice its width. If 22 feet of fencing is needed to enclose the garden, what are its dimensions?

20. On the first five tests you have scores of 61, 95, 71, 83, and 80. The last test, a final exam, counts as two grades. What score do you need on the final in order to have an average score of 80?